Prof. B. Pierce
Harvard University
with best regards of
The Author

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RECOMMENDATIONS.

Extract of a letter from Hon. John C. Spencer, Secretary of State, and one of the Regents of the University of the State of New York.

ALBANY, June 26, 1841.

MESSRS. BENNETT, BACKUS, & HAWLEY:

GENTLEMEN—The Committee of the Regents of the University, have selected "PERKINS' ARITHMETIC" to be purchased and used in the departments for the instruction of Common School teachers, established by the Regents, in several Academies of the State.

JOHN C. SPENCER.

From Hon. John A. Dix, one of the Regents of the University of the State of New York.

ALBANY, June 8, 1841.

MESSRS. BENNETT, BACKUS, & HAWLEY:

GENTLEMEN—I have examined Mr. George R. Perkins' Higher Arithmetic, and consider it a very valuable work, filling as it does, a space which is usually but a portion of more voluminous treatises. For academies and high schools, and for the purpose of exercise and discipline to those who have become familiar with the common rules of arithmetic, it is appropriate and useful, both on account of the manner in which the various subjects are treated, and the clearness of its illustrations.

JOHN A. DIX.

From I. W. Jackson, A. M., Professor of Mathematics, Union College.

SCHENECTADY, June 4, 1841.

MESSRS. BENNETT, BACKUS, & HAWLEY:

GENTLEMEN—I have examined the treatise on the Higher Arithmetic, by Prof. Perkins, which you were so good as to send me, and am most happy to give you my opinion of its merits.

It is a work of an order superior to any that has issued from the American press on the subject, in the highest degree creditable to the professor, and to the science of the country. Indeed, I am acquainted with no work on arithmetic in the English language equal to it. I am confident that its general adoption as a text-book, by our seminaries, would be considered by all who feel an interest in the promotion of the exact sciences, as an omen of good.

In great haste, your ob't serv't, I. W. Jackson.

From B. Birdsall, Professor of Mathematics, Clinton Liberal Institute.

Clinton, May 28, 1841.

Messrs. Bennett, Backus, & Hawley:

Gentlemen—I have examined with considerable attention "Perkins' Higher Arithmetic," and am favorably impressed with its excellence. It has been introduced here as one of the text-books for the use of those students who have already a knowledge of elementary arithmetic. The author is very clear in his explanations, systematic in the arrangement, and the examples are numerous and well-selected. In the explanations which generally precede the rules, the student is carried on by a kind of inductive process, to a perception of the principle on which the rule is founded. The article on continued fractions—which are so useful in determining approximate ratios—supplies a deficiency which exists in all *other* systems of arithmetic with which I am acquainted. In short, there are many new and interesting properties of numbers exhibited in the body of the work, which will, I believe, render it acceptable even to advanced students. The new and concise methods which are given for the extraction of the cube and other higher roots, will of themselves sufficiently commend the work to any one who is favorable to the improvement of science. B. Birdsall.

I concur in the above recommendation of Perkins' Higher Arithmetic. Timothy Clowes, Principal Clinton Liberal Institute.

From O. Root, A. M., Principal of the Syracuse Academy.

Syracuse, May 26, 1841.

Messrs. Bennett, Backus, & Hawley:

Gentlemen—I thank you for the copy of Prof. Perkins' Arithmetic which you had the kindness to send me. My engagements have prevented me from examining the work very minutely, but from what I have read I am satisfied that it will prove highly useful to those students

who pursue the study of arithmetic beyond our elementary treatises. I noticed several remarkable properties of numbers calculated to excite the curiosity and exercise the ingenuity of the student, and the exposure of a single defective rule which had so long remained uncorrected should entitle the work to the consideration of all teachers and lovers of the exact sciences. I shall commence an advanced class with the work as soon as it can be obtained. Yours, respectfully, O. ROOT.

From Marcus Catlin, A. M., A. A. S., Professor of Mathematics and Astronomy, Hamilton College.

CLINTON, May 25, 1841.

MESSRS. BENNETT, BACKUS, & HAWLEY:

GENTLEMEN—I have received a copy of Perkins' Higher Arithmetic which you had the kindness to forward to me. I had often expressed my approbation of the plan to the author while he was preparing the manuscript; and now that I have given the work a somewhat thorough examination, I am happy to say not only that my former opinion of its plan is confirmed, but that the plan has been well executed. This work appears to me to supply a deficiency in the series of text-books now in use; and if it would be generally adopted and introduced into our schools, it would undoubtedly tend to elevate the standard of arithmetical learning which is now unfortunately so low.

Yours, &c. MARCUS CATLIN.

From J. T. Foster, Principal of the High School, Little Falls.

Upon a careful comparison of Perkins' Higher Arithmetic with others, I am prepared to give it a high place among the books which have been published upon the subject of mathematics.

It seems admirably calculated to fill the vacancy which has existed between common arithmetic and algebra. In my opinion it is a work of peculiar excellence. The author has developed many new principles, which have never before appeared in any similar work. Had the author done nothing more than correct the rule for finding the "least common multiple of numbers," that alone would entitle the work to a large share of public patronage.

The method of explaining and developing the principles contained in the work is so clear and concise, that any scholar of ordinary abilities will find no difficulty in readily understanding them. During several years' experience in teaching the higher, as well as the ordinary branches of mathematics, I have found a vacancy which I have endeavored to fill up with oral instruction, without any definite system, till some one emu-

lous to confer a public benefit, should supply the desideratum. Although the work is not designed immediately for primary schools, still every teacher should have a copy of it for his own benefit. I should go on to particularize the excellencies of the work, but, after specifying the "explanation of prime numbers," the rules for discounting bank notes, permutation, &c., I shall leave its merits to its only true tests, of time and use.

J. T. Foster.

From Stephen W. Taylor, Professor of Mathematics and Natural Philosophy in Hamilton Literary and Theological Seminary.

Hamilton, May 29, 1841.

Messrs. Bennett, Backus, & Hawley:

Gentlemen—As a lover of learning, and a friend to youth, I feel indebted to any man, who takes pains to give to our schools an improved arithmetic.

From reviewing "Higher Arithmetic, by Geo. R. Perkins, A. M.," I am, therefore, happy to find in it, the plan judicious, the principles clearly developed, the introduction of the rules natural and easy, and the enunciation simple and neat. In my opinion, this book, in comparison with most others extant on the same subject, will serve to *advance* the student, in the *indispensable branch of Mathematics.*

Very respectfully, yours, Stephen W. Taylor.

From Alvin Lathrop, A. M., Professor of Mathematics in Pokeepsie Collegiate School.

Pokeepsie, June 4, 1841.

Messrs. Bennett, Backus, & Hawley:

Gentlemen—I received from you some weeks since a copy of Prof. Perkins' Higher Arithmetic, and have given it sufficient attention to satisfy me that it is a work of much ability; furnishing abundant evidence of having come from a mind much accustomed to arithmetical investigations, and familiar, in no common degree, with the properties and powers of numbers, and their manifold curious applications. I hope the author may find cheering encouragement and reward for his labors in the approbation of the public and in improved methods of instruction, under his guidance, in the higher department of arithmetic; a branch of education which, as he justly says, lies at the foundation of all excellence in mathematical sciences.

Very respectfully, Alvin Lathrop.

HIGHER ARITHMETIC,

DESIGNED FOR THE USE

OF

HIGH SCHOOLS, ACADEMIES, AND COLLEGES;

IN WHICH

SOME ENTIRELY NEW PRINCIPLES ARE DEVELOPED, AND MANY CONCISE AND EASY RULES GIVEN, WHICH HAVE NEVER BEFORE APPEARED IN ANY ARITHMETIC.

By GEO. R. PERKINS, A. M.,

PRINCIPAL, AND PROFESSOR OF MATHEMATICS, UTICA ACADEMY.

PUBLISHED BY

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1841.

BENNETT, BACKUS, & HAWLEY,
PRINTERS, PUBLISHERS, BOOK-SELLERS, ETC.,
Franklin Square, Utica, N. Y.

PREFACE.

ARITHMETIC is a subject of so much importance, being that branch of mathematics upon which all the others are based, that any attempts to elucidate its rules of operation must be considered as worthy of commendation, even should those endeavors fail of their object.

I am well aware that our schools are already flooded with books on this subject, but among the multiplicity of works which have appeared within a few years, there seems not to have been any material change; they all wear nearly the same aspect. Whilst all other school books have been rapidly improving, our Arithmetic has remained nearly stationary.

This work is not designed to teach the fundamental, or ground rules, of the science, but is intended for such pupils as have already pursued some simple elementary Arithmetic as far as the *Rule of Three.* I know of no *elementary* work better adapted for this purpose, than "DAVIES' MENTAL AND PRACTICAL ARITHMETIC."

I have endeavored to simplify many of those rules, which hitherto have been considered as the most difficult.

Under Chapter I., will be found many important properties of numbers, demonstrated by the aid of *prime numbers;* under this Chapter will also be found an exposure of the *erroneous rule*, given in nearly all our Arithmetics, for finding the least common multiple of several numbers.

Under Chapter III., will be found some entirely new things in reference to that class of repetends, which I denominate ***Perfect Repetends.***

The rule for extracting the ***cube root***, as well as the general rule for roots of all powers, as given under Chapter XI., has been drawn from Mr. Holdred's method of solving Algebraic Equations; which was first published in 1820. I am also indebted, for the arrangement of the numerical work, to the "***Root Extractor;***" a work in pamphlet form, by Timothy Clowes, LL. D., published in 1831. These rules have been universally admired by all who have used them.

Under Chapter XV., I have given Analytical Solutions to that class of questions, which, by most authors, are solved by ***Position;*** which rule, in my opinion, should never be used when a direct solution can be obtained. This method of solving questions is preferable to every other, since it appeals to the reasoning powers of the mind, and is not, like many arithmetical rules, a mere mechanical method of operation.

Many of these questions have been made expressly for this work, others have been copied from standard works on *Algebra.*

GEO. R. PERKINS.

Utica, April, 1841.

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ARITHMETIC.

CHAPTER I.

DEFINITIONS.

1. Any whole number is called an *integer*.

2. Any number which can be divided by 2, without a remainder, is called an *even number*.

3. All numbers which cannot be divided by 2, without a remainder, are called *odd numbers*.

4. Any number which can be produced by multiplying two or more numbers together, each of which is greater than a unit, is called a *composite number*. Thus, 35 is a composite number, since it can be produced by multiplying 5 and 7 together.

5. The numbers which are multiplied together to produce a composite number, are called *factors*. Thus, 3 and 8 are factors of 24; so also are 4 and 6.

6. A composite number which is composed of two equal factors, is called a *square number*. Thus, 4, 9, 16, and 49, are square numbers.

7. A composite number which is composed of three equal factors, is called a *cube number*. Thus, 8, 27, and 64, are cube numbers.

8. One of the equal factors which compose a square number, is called the *square root* of the number. Thus, 7 is called the square root of 49.

9. One of the equal factors which compose a cube number is called the *cube root* of the number. Thus, 3 is the cube root of 27.

10. All numbers which are not composite, are called *prime numbers*. Thus, 1, 2, 3, 5, 7, 11, and 13, are prime numbers.

11. Unity divided by a number is the *reciprocal* of that number.

SYMBOLS.

2. The symbol, $=$, is called the sign of *equality;* and denotes that the quantities between which it is placed are equal or equivalent to each other. Thus, $\$1=100$ cents, which is read, one dollar equals one hundred cents.

2. The symbol, $+$, is called *plus*; and denotes that the quantities between which it is placed are to be added together. Thus, $6+2=8$, which is read, six and two added equals eight.

3. The symbol, $-$, is called *minus;* and denotes that the quantity which is placed at the right of it is to be subtracted from the quantity on the left. Thus, $6-2=4$, which is read, six diminished by two equals four.

4. The symbol, $\times$, is called the sign of *multiplication;* and denotes that the quantities between which it is placed are to be multiplied together. Thus, $6\times2=12$, which is read, six multiplied by two equals twelve.

5. The symbol, $\div$, is called the sign of *division;* and denotes that the quantity on the left of it is to be divided by the quantity on the right. Thus, $6\div2=3$, which is read, six divided by two equals three.

Division is also denoted by placing the divisor under the dividend, with a horizontal line between them like a vulgar fraction. Thus, $\frac{6}{2}$ is the same as $6 \div 2$.

6. A number placed above another number, a little to the right, is called an *exponent*. Thus, 6^2, 7^3, in these expressions, 2 and 3 are exponents of 6 and 7 respectively.

7. An exponent placed over a quantity denotes that the quantity is to be used as a factor as many times as there are units in the exponent. Thus, $2^4 = 2 \times 2 \times 2 \times 2 = 16$.

8. When the exponent is two, the result is called the *second power* of the quantity over which it is placed. Thus, $7^2 = 7 \times 7 = 49 =$ the second power of 7.

9. When the exponent is three, the result is called the *third power* of the quantity over which it is placed. Thus, $4^3 = 4 \times 4 \times 4 = 64 =$ the third power of 4.

The higher powers are denoted in the same way.

10. The symbol, $\sqrt{\ }$, denotes that the *square root* of the quantity over which it is placed is to be taken. Thus, $\sqrt{4} = 2$, which is read, the square root of 4 equals 2.

11. The symbol, $\sqrt[3]{\ }$, denotes in a similar manner the *cube root* of the number over which it is placed. Thus, $\sqrt[3]{64}$, $= 4$ which is read, the cube root of 64 equals 4.

The roots of higher dimensions are denoted in a similar way.

12. The symbol, $\therefore$, is equivalent to the phrase *therefore*, or *consequently*. Thus, $6^2 = 36$, and $4 \times 9 = 36$ $\therefore$ $6^2 = 4 \times 9$, which is read, the square of 6 equals 36, and the product of 4 and 9 equals 36, therefore the square of 6 equals the product of 4 and 9.

13. The parenthesis, ( ), when it encloses several quantities, requires these quantities to be regarded as one single quantity. Thus, $(5+2)\times7=49$, which is read, the sum of 5 and 2 multiplied by $7=49$.

EXAMPLES.

ILLUSTRATING THE FOREGOING DEFINITIONS AND SYMBOLS.

1. The expression, $11+5-2=2\times7=28\div2=14$, when translated into common language, becomes the sum of 11 and 5 diminished by 2 equals the product of 2 and 7, equals 28 divided by 2, equals 14.

2. The expression, $\frac{(42-8)}{2}+3=4\times5=20$, is equivalent to the following: Forty two diminished by 8, and the remainder divided by 2, and the quotient increased by 3, equals 4 multiplied into 5, equals 20.

3. The expression, $\sqrt{144}=3\times4=36\div3=12$, is the same as the square root of 144 equals 3 multiplied into 4, equals 36 divided by 3, equals 12.

4. Translate the expression, $(10+3)\times7=182\div2=91$, into common language.

5. Translate the expression, $(\sqrt{16}+7)\times4=\sqrt[3]{64}\times11$, into common language.

6. Translate the expression, $(\sqrt{49}-\sqrt[3]{64})\times3=20-11$, into common language.

7. What expression is equivalent to the following? "Five times nine divided by three, and that quotient multiplied by seven, equals the square of ten increased by five?"

Ans. $\frac{5\times9}{3}\times7=10^2+5$

8. What expression is equivalent to the following ?—"Three times twenty-one, increased by five times seven, and diminished by three times the square of four, is equal to twice the square of five ?"

9. What expression is equivalent to the following? "The cube root of sixty-four, increased by two, and the sum multiplied by ten, is equal to the square of eight diminished by four?"

MULTIPLICATION OF COMPOUND EXPRESSIONS.

4. Let it be required to multiply $3+2$ by $4+5$.

We must repeat $3+2$ as many times as there are units in $4+5$.

First, repeating $3+2$ as many times as there are units in 4, we get $(3+2)\times 4=12+8$, for first partial product:

Secondly, repeating $3+2$ as many times as there are units in 5, we get $(3+2)\times 5=15+10$, for second partial product:

Hence, $3+2$ repeated as many times as there are units in $4+5$, becomes $(3+2)\times(4+5)=12+8+15+10$.

2. Again, let it be required to multiply $7-3$ by $4+2$: Proceeding as in the last example, we find $(7-3)\times(4+2)=28-12+14-6$.

3. In a similar way we find that $4-3$, multiplied by $3-2$, gives $(4-3)\times(3-2)=12-9-8+6$.

By carefully reviewing these examples, we deduce the following rule. To multiply together two compound expressions.

RULE.

Multiply each term of one of the factors, by each term of the other factor; observing that like signs produce plus, and unlike signs produce minus.

Examples.

4. What is the product of $8+3$ by $6+4$?

Ans. $48+18+32+12$.

5. What is the product of $6-2$ by $4+3$?

Ans. $24-8+18-6$.

6. What is the product of $11-3$ by $13-7$?

Ans. $143-39-77+21$.

7. What is the product of $3+2-1$ by $4-1+5$?

Ans. $12+8-4-3-2+1+15+10-5$.

8. What is the product of $1+2-3$ by $4-5+6$?

Ans. $4+8-12-5-10+15+6+12-18$.

9. What is the product of $7-9$ by $5-11$?

10. What is the product of $21-3$ by $9-2$?

11. What is the product of $1+7+5$ by $2+3$?

12. What is the product of $9+8+7$ by $6+5+4$?

INTERESTING PROPERTIES OF NUMBERS.

PROPOSITION I.

5. Every number will divide by 9, when the sum of its digits is divisible by 9.

For, take any number as 78534; this number is, by the nature of decimal arithmetic, the same as $70000+8000+500+30+4$.

$$\begin{aligned} \text{Now } 70000 &= 9999\times7+7 \\ 8000 &= 999\times8+8 \\ 500 &= 99\times5+5 \\ 30 &= 9\times3+3 \\ 4 &= \quad +4 \end{aligned}$$

$\therefore\ 78534=9999\times7+999\times8+99\times5+9\times3+(7+8+5+3+4.)$

Now, since each expression, 9999×7, 999×8, 99×5, and 9×3, is divisible by 9, it follows that the first number, 78534, will be divisible by 9 when the sum of its digits $(7+8+5+3+4)$ is.

Hence it follows, that any number being diminished by the sum of its digits, will become divisible by 9.

Also any number divided by 9 will leave the same remainder as the sum of the digits when divided by 9.

NOTE.—These singular properties of the digit 9, have been made use of by many authors for proving the work of the four fundamental rules of arithmetic.

PROPOSITION II.

6. Every number is either a prime number, or composed of prime factors.

For, all numbers which are not prime are composite, and can therefore be separated into two or more factors; and if these factors are not prime, they can again be separated into other factors, and thus the decomposition can be continued until all the factors are prime.

Hence, to resolve any composite number into its prime factors, we have this

RULE.

Divide the number by any prime number, which will divide it without any remainder; then divide the quotient in the same way; and so continue until a quotient is obtained which is a prime. Then will the successive divisors, together with the last quotient, form the prime factors required.

Examples.

1. Resolve 728 into its prime factors.

$$\begin{array}{r|l} 2 & 728 \\ \hline 2 & 364 \\ \hline 2 & 182 \\ \hline 7 & 91 \\ \hline & 13 \end{array}$$

Therefore, $2\times2\times2\times7\times13=2^3\times7\times13$ are the prime factors of 728.

2. Resolve 812 into its prime factors.

Ans. $2^2\times7\times29$.

3. What are the prime factors of 978.

Ans. $2\times3\times163$.

4. What are the prime factors of 1011?

Ans. 3×337.

5. What are the prime factors of 100?

Ans. $2^2\times5^2$

6. What are the prime factors of 8975?

7. What are the prime factors of 808?

8. What are the prime factors of 707?

9. What are the prime factors of 1118?

10. What are the prime factors of 1098?

7. As we shall make so frequent use of prime numbers, we will give a table of some of the lowest primes.

TABLE OF PRIME NUMBERS.

1	131	311	503	719	941	1163	1423	1619	1877	2129
2	137	313	509	727	947	1171	1427	1621	1879	2131
3	139	317	521	733	953	1181	1429	1627	1889	2137
5	149	331	523	739	967	1187	1433	1637	1901	2141
7	151	337	541	743	971	1193	1439	1657	1907	2143
11	157	347	547	751	977	1201	1447	1663	1913	2153
13	163	349	557	757	983	1213	1451	1667	1931	2161
17	167	353	563	761	991	1217	1453	1669	1933	2179
19	173	359	569	769	997	1223	1459	1693	1949	2203
23	179	367	571	773	1009	1229	1471	1697	1951	2207
29	181	373	577	787	1013	1231	1481	1699	1973	2213
31	191	379	587	797	1019	1237	1483	1709	1979	2221
37	193	383	593	809	1021	1249	1487	1721	1987	2237
41	197	389	599	811	1031	1259	1489	1723	1993	2239
43	199	397	601	821	1033	1277	1493	1733	1997	2243
47	211	401	607	823	1039	1279	1499	1741	1999	2251
53	223	409	613	827	1049	1283	1511	1747	2003	2267
59	227	419	617	829	1051	1289	1523	1753	2011	2269
61	229	421	619	839	1061	1291	1531	1759	2017	2273
67	233	431	631	853	1063	1297	1543	1777	2027	2281
71	239	433	641	857	1069	1301	1549	1783	2029	2287
73	241	439	643	859	1087	1303	1553	1787	2039	2293
79	251	443	647	863	1091	1307	1559	1789	2053	2297
83	257	449	653	877	1093	1319	1567	1801	2063	2309
89	263	457	659	881	1097	1321	1571	1811	2069	2311
97	269	461	661	883	1103	1327	1579	1823	2081	2333
101	271	463	673	887	1109	1361	1583	1831	2083	2339
103	277	467	677	907	1117	1367	1597	1847	2087	2341
107	281	479	583	911	1123	1373	1601	1861	2089	2347
109	283	487	691	919	1129	1381	1607	1867	2099	2351
113	293	491	701	929	1151	1399	1609	1871	2111	2357
127	307	499	709	937	1153	1409	1613	1873	2113	2371

8. Suppose we wish to know whether the numbers 204 and 468 have a common factor; we proceed as follows: We decompose them into their prime factors, and thus obtain $204=2^2\times3\times17$, and $468=2^2\times3^2\times13$. Here we see that $2^2\times3$ is common to both the numbers 204 and 468.

Hence, to find the greatest factor which is common to two or more numbers, or, as generally expressed, to find the greatest common measure of two or more numbers, we have this

RULE.

Resolve the numbers into their prime factors, (by Rule under Art. 6.) Then select such of the primes as are common to all the numbers, multiply them together, and the product will give the greatest common measure.

Examples.

1. What is the greatest common measure of 1326, 3094 and 4420?

These numbers, when resolved into the prime factors, become

$$1326=2\times3\times13\times17$$
$$3094=2\times7\times13\times17$$
$$4420=2^2\times5\times13\times17$$

The factors which are common are 2, 13, and 17; therefore the greatest common measure is $2\times13\times17=442$.

2. What is the greatest common measure of 556, 672, and 840? Ans. $2^2=4$.

3. What is the greatest common measure of 110, 140, and 680? Ans. $2\times5=10$.

4. What is the greatest common measure of 255, and 532? Ans. They have none.

5. What is the greatest common measure of 375, 408, and 922? Ans. They have none.

9. We may also find the greatest common measure of two numbers by the following

RULE.

Divide the greater by the less, then divide the divisor by the remainder, and thus continue to divide the preceding divisor by the last remainder, until there is no remainder. The last divisor will be the greatest common measure.

Examples.

1. What is the greatest common measure of 360, and 630 ?

OPERATION.

```
360)630(1
    360
    ---
    270)360(1
        270
        ---
         90)270(3
            270
            ---
              0
```

Hence the greatest common measure is 90.

2. What is the greatest common measure of 922, and 408? Ans. 2.

3. What is the greatest common measure of 1825, and 2555? Ans. 365.

4. What is the greatest common measure of 124, and 682? Ans. 62.

5. What is the greatest common measure of 296, and 407? Ans. 37.

6. What is the greatest common measure of 404, and 364? Ans. 4.

7. What is the greatest commmon measure of 506, and 308?

8. What is the greatest common measure of 212, and 416?

9. What is the greatest common measure of 74, and 84?

10. Suppose we wish to know what is the least number which will divide by 215 and 460; we proceed as follows: We decompose them into their prime factors, and thus obtain $215=5\times43$, $460=2^2\times5\times23$. Hence, we see that $2^2\times5\times23\times43=19780$, is the least number which can be divided by 215 and 460.

Hence, to find the least number which will divide by two or more numbers, or as generally expressed, to find the least common multiple, we have this

RULE.

Resolve the numbers into their prime factors (by Rule under Art. 6,) Select all the different factors which occur, observing that, when the same factor has different powers, to take the highest power. The continued product of the factors thus selected will give the least common multiple.

Examples.

1. What is the least common multiple of 12, 16, and 24?

These numbers resolved into their prime factors give

$$12=2^2\times3$$
$$16=2^4$$
$$24=2^3\times3$$

Therefore $2^4\times3=48$ is the least multiple required.

2. What is the least common multiple of 9, 12, 16, 20, and 35? Ans. 5040.

5. What is the least common multiple of 7, 13, 39, and 84? Ans. 1092.

4. What is the least common multiple of the nine digits? Ans. 2520.

5. What is the least common multiple of 3, 5, 7, 12, 15, 18, and 35? Ans. 1260.

6. What is the least common multiple of 100, 109, 463, and 900?

7. What is the least common multiple of 365, 910, 2217, and 2424?

11. We may also find the least common multiple of two or more numbers by the following

RULE.*

Write the numbers in a horizontal line, divide them by any prime number, which will divide two or more of them, place the quotients with the undivided terms for a second horizontal line, proceed with this second line as with the first; and so continue until there are no two terms which can be divided. The

* This rule is usually given as follows: "Write down the numbers in a line, and divide them by any number that will measure two or more of them; and write the quotients and undivided numbers in a line beneath. Divide this line as before, and so on, until there are no two numbers that can be measured by the same divisor; then the continual product of all the divisors and numbers in the last line will be the least common multiple required."

The above we have copied from Mr. Adams' Arithmetic; nearly all our arithmetics give in substance the same rule. We will now show by an example that this rule may give very different results, depending upon the divisors used, and of course the rule is in fault.

EXAMPLE.

What is the least common multiple of 12, 16, and 24? We will work this example in three ways, as follows.

1st OPERATION.		2d OPERATION.		3d OPERATION.	
12	12, 16, 24	8	12, 16, 24	4	12, 16, 24
2	1, 16, 2	3	12, 2, 3	3	3, 4, 6
	1, 8, 1	2	4, 2, 1	2	1, 4, 2
			2, 1, 1		1, 2, 1

$12 \times 2 \times 8 = 192$ $\quad 8 \times 3 \times 2 \times 2 = 96$ $\quad 4 \times 3 \times 2 \times 2 = 48$

These operations, which are wrought strictly by this rule, give 192, 96, and 48 for the least multiple of 12, 16, and 24. Hence the rule is wrong, and can not be depended upon. The least common multiple of 12, 16, and 24, is truly 48, as may be found by either of our rules.

continued product of the divisors and the numbers in the last horizontal line will give the least common multiple.

Examples.

1. What is the least common multiple of 28, 35, 42, 77, and 70?

OPERATION.

$$\begin{array}{r|l} 7 & 28,\ 35,\ 42,\ 77,\ 70. \\ \hline 5 & 4,\ \ 5,\ \ 6,\ 11,\ 10. \\ \hline 2 & 4,\ \ 1,\ \ 6,\ 11,\ \ 2. \\ \hline & 2,\ \ 1,\ \ 3,\ 11,\ \ 1. \end{array}$$

Hence, $7\times5\times2\times2\times3\times11=4620$, is the multiple sought.

2. What is the least common multiple of 46, 92, 374, and 23? Ans. 17204.

3. What is the least common multiple of 5, 15, 36, and 72? Ans. 360.

4. What is the least common multiple of 11, 77, 88, and 92?

5. What is the least common multiple of 14, 51, 102, and 500?

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12. Suppose we wish to find all the divisors of 36, we proceed as follows: We resolve 36 into its prime factors, and thus obtain $36=2^2\times3^2$.

Now it is obvious, that any combination of 2 and 3, which does not make use of these factors in a higher power than they occur in $2^2\times3^2$ must be a divisor of 36. All such combinations can be found by multiplying $1+2+4$ by $1+3+9$, performing this multiplication we obtain

$$\begin{array}{l} 1+2+4 \\ \underline{1+3+9} \\ 1+2+4+3+6+12+9+18+36. \end{array}$$

Therefore, the divisors of 36 are 1, 2, 4, 3, 6, 12, 9, 18, and 36.

Hence, to find all the divisors of any number, we have this

RULE.

Resolve the number into its prime factors, form as many series of terms as there are prime factors, by making 1 *the first term of any one of the series, the first power of one of the prime factors for the second term, the second power of this factor for the third term, and so on, until we reach a power as high as occurred in the decomposition. Then multiply these series together, (by rule under Art. 4,) and the partial products thus obtained will be the divisors sought.*

Examples.

1. What are the divisors of 48?

Here we find $48=2^4\times3$. Therefore our series of terms will be $1+2+4+8+16$ and $1+3$, multiplying these together (by rule under Art. 4,) we get

$$\begin{array}{l} 1+2+4+8+16 \\ \underline{1+3} \\ 1+2+4+8+16+3+6+12+24+48. \end{array}$$

Therefore, the divisors of 48 are 1, 2, 4, 8, 16, 3, 6, 12, 24, and 48.

2. What are the divisors of 360?

Ans. $\begin{cases} 1, 2, 4, 8, 3, 6, 12, 24, 9, 18, 36, 72, \\ 5, 10, 20, 40, 15, 30, 60, 120, 45, 90, 180, 360. \end{cases}$

3. What are the divisors of 100?

Ans. 1, 2, 4, 5, 10, 20, 25, 50, 100.

4. What are the divisors of 810?

5. What are the divisors of 920?

6. What are the divisors of 840?

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13. Since the series of terms, which we multiplied together by the last rule, to obtain the divisors of any number, commenced with 1, it follows that the number of terms in each series will be one more than the units in the exponent of the factor used.

Hence, to find the number of divisors of any number, without exhibiting them, we have this

RULE.

Resolve the number into its prime factors, increase the exponents by a unit, and then take their continued product, and it will express the number of divisors.

Example.

1. How many divisors has 4320?

$4320=2^5\times3^3\times5$. In this case the exponents are 5, 3, and 1, each of which being increased by one, we obtain 6, 4, and 2, the continued product of which is $6\times4\times2=48$, the number of divisors sought.

2. How many divisors has 300? Ans. 18.

3. How many divisors has 3500? Ans. 24.

4. How many divisors has 162000? Ans. 100.

5. How many divisors has 824?

6. How many divisors has 1172?

7. How many divisors has 6336?

CHAPTER II.

FRACTIONS.

14. A fraction is an expression representing a part of a unit.

VULGAR FRACTIONS.

15. A vulgar fraction consists of two numbers, the one placed above the other as in division.

The number above the line is called the *numerator*; the number below the line is called the *denominator*.

Thus, $\frac{5}{8}$ is a vulgar fraction, whose numerator is 5, and whose denominator is 8: it is read *five-eighths*.

The denominator shows how many parts the unit is divided into; and the numerator shows how many of these parts are used.

Thus, 5/8 denotes that the unit is divided into 8 equal parts, and that 5 of these parts are used.

When the numerator is equal to the denominator the fraction is equivalent to a unit. Thus, $\frac{6}{6}$, $\frac{11}{11}$, $\frac{5}{5}$, and $\frac{31}{31}$, are each equivalent to 1.

When the numerator is less than the denominator, the value of the fraction is less than a unit; it is then called a *proper fraction*.

Thus, $\frac{5}{8}$, $\frac{3}{7}$, $\frac{10}{41}$, and $\frac{20}{95}$, are each proper fractions.

When the numerator is larger than the denominator, its value is then more than a unit; it is therefore called an *improper fraction*.

Thus, $\frac{8}{3}$, $\frac{7}{2}$, $\frac{13}{7}$, and $\frac{40}{29}$, are each improper fractions.

A fraction of a fraction, connected by the word *of*, is called a compound fraction.

Thus, $\frac{2}{3}$ of $\frac{5}{8}$ of $\frac{2}{11}$ of $\frac{7}{6}$, $\frac{1}{2}$ of $\frac{2}{3}$ of $\frac{6}{7}$, and $\frac{3}{11}$ of $\frac{11}{13}$ of $\frac{2}{3}$, are compound fractions.

A fraction is said to be *inverted*, when the numerator and the denominator change places.

Thus, the fractions $\frac{16}{17}$, $\frac{11}{5}$, and $\frac{5}{8}$, when inverted, become $\frac{17}{16}$, $\frac{5}{11}$, and $\frac{8}{5}$.

Any integer may take the form of an improper fraction, by writing a unit for its denominator.

Thus, 6, 5, 3, and 11, are the same as the improper fractions $\frac{6}{1}$, $\frac{5}{1}$, $\frac{3}{1}$, and $\frac{11}{1}$.

A number consisting of an integer and fraction, is called a *mixed number*.

Thus, $4\frac{1}{2}$, $5\frac{7}{9}$, $6\frac{3}{8}$, and $13\frac{11}{17}$, are mixed numbers. They may also be written $4+\frac{1}{2}$, $5+\frac{7}{9}$, $6+\frac{3}{8}$, and $13+\frac{11}{17}$.

REDUCTION OF FRACTIONS.

16. Since the value of a fraction is the quotient arising from dividing the numerator by its denominator, we may infer the following propositions.

I. That, multiplying the numerator of a fraction by any number, is the same as multiplying the value of the fraction by the same number.

II. That, multiplying the denominator of a fraction by any number, is the same as dividing the value of the fraction by the same number.

III. That, multiplying both numerator and denominator by the same number, does not alter the value of the fraction.

IV. That, dividing the numerator of a fraction by any number, is the same as dividing the value of the fraction by the same number.

V. That, dividing the denominator of a fraction by any number, is the same as multiplying the value of the fraction by the same number.

VI. That, dividing both numerator and denominator by any number, does not alter the value of the fraction.

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17. When the numerator and denominator of a fraction have no common measure, it is said to be in its *lowest terms.*

To reduce simple fractions to their lowest terms, we have the following

RULE.

Divide both numerator and denominator by their greatest common measure, (found by one of the rules under Art. 8, *or* 9.) *This division will not alter the value of the fraction.* (*Prop. VI. Art.* 16.)

Examples.

1. Reduce $\frac{375}{425}$ to its lowest terms.

In this example we find the greatest common measure of 375 and 425, to be 25.

Dividing both numerator and denominator by 25, we find $\frac{375}{425}=\frac{15}{17}$.

2. Reduce $\frac{1049}{8392}$ to its lowest terms. Ans. $\frac{1}{8}$.

3. Reduce $\frac{172}{1118}$ to its lowest terms. Ans. $\frac{2}{13}$.

4. Reduce $\frac{1740}{2900}$ to its lowest terms. Ans. $\frac{3}{5}$.

5. Reduce [illegible] to its lowest terms. Ans. [illegible].

6. Reduce $\frac{20012}{37956}$ to its lowest terms.

7. Reduce $\frac{432}{3080}$ to its lowest terms.

8. Reduce $\frac{448}{3066}$ to its lowest terms.

9. Reduce $\frac{3455}{4235}$ to its lowest terms.

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18. To reduce an improper fraction to a mixed number, we have this

RULE.

Divide the numerator by the denominator, the quotient will be the integral part of the mixed number. The remainder placed over the denominator of the improper fraction, will form the fractional part.

The correctness of the above rule is obvious from considering that the value of a fraction is the quotient arising from dividing the numerator by the denominator.

Examples.

1. Reduce $\frac{17}{4}$ to a mixed number.

Dividing 17 by 4, we obtain the quotient 4, with the remainder 1; ∴ the mixed number equivalent to $\frac{17}{4}$ is $4\frac{1}{4}$ or $4+\frac{1}{4}$.

2. What mixed number is equivalent to $\frac{131}{7}$?

Ans. $18\frac{5}{7}$.

3. What mixed number is equivalent to $\frac{12345}{11}$?

Ans. $1122\frac{3}{11}$.

4. What mixed number is equivalent to $\frac{29817}{365}$?

Ans. $81\frac{252}{365}$.

5. What mixed number is equivalent to $\frac{70063}{6092}$?

6. What mixed number is equivalent to $\frac{3729}{302}$?

7. What mixed number is equivalent to $\frac{2368}{37}$?

8. What mixed number is equivalent to $\frac{48597354}{37538}$?

9. What mixed number is equivalent to $\frac{227229}{433}$?

10. What mixed number is equivalent to $\frac{226072}{4583}$?

11. What mixed number is equivalent to $\frac{36573}{4927}$?

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19. To reduce a mixed number to its equivalent improper fraction, we have this

RULE.

Multiply the integral part of the mixed number by the denominator of the fractional part, to the product add the numerator of the fractional part, the sum will be the numerator of the improper fraction; under which place the denominator of the fractional part.

This rule is obviously correct, since it is the reverse of the rule under Art. 18, where a reverse operation was required to be performed.

Examples.

1. Reduce $13\frac{6}{7}$ to an improper fraction.

Multiplying the integer 13 by the denominator 7, we obtain 91; to which, adding the numerator 6, we get 97 for the numerator of the improper fraction; $\therefore$ the improper fraction equivalent to $13\frac{6}{7}$ is $\frac{97}{7}$.

2. What improper fraction is equivalent to $1278\frac{1}{3}$?

Ans. $\frac{3835}{3}$.

3. What improper fraction is equivalent to $18910\frac{4}{7}$?

Ans. $\frac{132374}{7}$.

4. What improper fraction is equivalent to $492536\frac{11}{13}$?

Ans. $\frac{6402979}{13}$.

5. What improper fraction is equivalent to $5\frac{2729}{38783}$?
6. What improper fraction is equivalent to $11\frac{245}{3223}$?
7. What improper fraction is equivalent to $23\frac{27}{2329}$?
8. What improper fraction is equivalent to $875\frac{6}{773}$?
9. What improper fraction is equivalent to $6833\frac{7453}{44643}$?
10. What improper fraction is equivalent to $11223344\frac{5}{11}$?

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20. Reduce the compound fraction $\frac{3}{4}$ of $\frac{7}{11}$ to its equivalent simple fraction.

$\frac{1}{4}$ of $\frac{7}{11}$ can be obtained by dividing the value of the fraction $\frac{7}{11}$ by 4, which (by Prop. II., Art. 16) can be effected by multiplying the denominator by 4; $\therefore \frac{1}{4}$ of $\frac{7}{11} = \frac{7}{4 \times 11}$.

Again $\frac{3}{4}$ of $\frac{7}{11}$ is obviously three times as great as $\frac{1}{4}$ of $\frac{7}{11}$; $\therefore$ to obtain $\frac{3}{4}$ of $\frac{7}{11}$ we must multiply $\frac{7}{4 \times 11}$ by 3, which (by Prop. I. Art. 16) can be done by multiplying the numerator by 3; hence, we have $\frac{3}{4}$ of $\frac{7}{11} = \frac{3 \times 7}{4 \times 11} = \frac{21}{44}$.

Hence, to reduce compound fractions to their equivalent simple ones we have this

RULE.

Consider the word OF, *which connects the fractional parts, as equivalent to the sign of multiplication. Then multiply all the numerators together for a new numerator; and all the denominators together for a new denominator. Always observing to reject, or cancel, such factors as are common to the numerators and denominators; which is the same as dividing both numerator and denominator by the same quantity, and (by Rule under Art. 17) does not change the value of the fraction.*

Examples.

1. Reduce $\frac{1}{2}$ of $\frac{3}{4}$ of $\frac{8}{15}$ of $\frac{5}{12}$ to its equivalent simple fraction.

Substituting the sign of multiplication for the word *of*, we get $\frac{1}{2}\times\frac{3}{4}\times\frac{8}{15}\times\frac{5}{12}$. First, cancelling the 8 of the numerator against the 2 and 4 of the denominator, by drawing a line across them, we get $\frac{1}{\cancel{2}}\times\frac{3}{\cancel{4}}\times\frac{\cancel{8}}{15}\times\frac{5}{12}$. Again, cancelling the 3 and 5 of the numerator against the 15 of the denominator, we finally obtain $\frac{1}{\cancel{2}}\times\frac{\cancel{3}}{\cancel{4}}\times\frac{\cancel{8}}{\cancel{15}}\times\frac{\cancel{5}}{12}=\frac{1}{12}$.

2. Reduce $\frac{3}{7}$ of $\frac{14}{35}$ of $\frac{7}{8}$ of $\frac{4}{9}$ of $\frac{5}{11}$ to its simplest form.

First, cancelling the 7 and 5 of the numerator against 35 of the denominator, we get $\frac{3}{7}\times\frac{14}{\cancel{35}}\times\frac{\cancel{7}}{8}\times\frac{4}{9}\times\frac{\cancel{5}}{11}$. Again, cancelling the 7 of the denominator against a part of the 14 of the numerator, and the 3 of the numerator against a part of the 9 of the denominator, we obtain $\frac{\cancel{3}}{\cancel{7}}\times\frac{\overset{2}{\cancel{14}}}{\cancel{35}}\times\frac{\cancel{7}}{8}\times\frac{4}{\underset{3}{\cancel{9}}}\times\frac{\cancel{5}}{11}$.

Finally, cancelling the 2 and 4 of the numerators against 8 of the denominator, we get $\frac{\cancel{3}}{\cancel{7}}\times\frac{\overset{\cancel{2}}{\cancel{14}}}{\cancel{35}}\times\frac{\cancel{7}}{\cancel{8}}\times\frac{\cancel{4}}{\underset{3}{\cancel{9}}}\times\frac{\cancel{5}}{11}=\frac{1}{33}$.

NOTE.—We have written our fractions several times, in order the more clearly to exhibit the process of cancelling. But in practice it will not be necessary to write the fraction more than once. It will make no difference which of the factors are first cancelled; when all the common factors have in this way been stricken out, the fraction will then appear in its lowest terms.

The student will find it to his interest to perform many examples of this kind, as this principle of cancelling will be extensively employed in the succeeding parts of this work.

3. Reduce $\frac{3}{11}$ of $\frac{9}{13}$ of $\frac{44}{49}$ of $\frac{22}{36}$ of $\frac{3}{10}$ to its simplest form.
Ans. $\frac{27}{988}$.

4. Reduce $\frac{1}{7}$ of $\frac{7}{8}$ of $\frac{16}{15}$ of $\frac{21}{4}$ of $\frac{8}{9}$ of $\frac{4}{11}$ of $\frac{4}{1}$ to its simplest form. Ans. $\frac{16}{11}$.

5. Reduce $\frac{3}{24}$ of $\frac{2}{9}$ of $\frac{14}{7}$ of $\frac{14}{29}$ of $\frac{4}{15}$ to its simplest form.
Ans. $\frac{2}{261}$.

6. Reduce $\frac{11}{23}$ of $\frac{39}{70}$ of $\frac{7}{9}$ of $\frac{39}{93}$ to its simplest form.

7. Reduce $\frac{1}{2}$ of $\frac{2}{6}$ of $\frac{3}{8}$ of $\frac{5}{22}$ of $\frac{11}{23}$ to its simplest form.

8. Reduce $\frac{1}{2}$ of $\frac{2}{3}$ of $\frac{3}{4}$ of $\frac{4}{5}$ of $\frac{5}{6}$ of $\frac{6}{7}$ of $\frac{7}{8}$ of $\frac{8}{9}$ to its simplest form.

9. Reduce $\frac{7}{23}$ of $\frac{11}{8}$ of $\frac{26}{55}$ of $\frac{19}{22}$ to its simplest form.

10. Reduce $\frac{3}{7}$ of $\frac{14}{25}$ of $\frac{19}{27}$ of $\frac{34}{35}$ to its simplest form.

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21. To reduce fractions to a common denominator, we have this

RULE.

Reduce mixed numbers to improper fractions, compound fractions to their simplest form. Then multiply each numerator by all the denominators, except its own, for a new numerator; and all the denominators together for a common denominator.

It is obvious that this process will give the same denominator to each fraction, viz: the product of all the denominators.

It is also obvious, that the values of the fractions will not be changed, since both numerator and denominator are multiplied by the same quantity, viz: the product of all the denominators except its own.

Examples.

1. Reduce $\frac{1}{2}$, $\frac{5}{3}$ of $\frac{2}{5}$, $\frac{3}{11}$, and $\frac{7}{9}$ of $\frac{3}{4}$ to equivalent fractions, having a common denominator.

These fractions when reduced to their simplest form are $\frac{1}{2}$, $\frac{2}{3}$, $\frac{3}{11}$, and $\frac{2}{9}$.

The new numerator of first fraction is $1 \times 3 \times 11 \times 9 = 297$.

" " " second fraction is $2 \times 2 \times 11 \times 9 = 396$.

" " " third fraction is $3 \times 2 \times 3 \times 9 = 162$.

" " " fourth fraction is $2 \times 2 \times 3 \times 11 = 132$.

The common denominator is $2 \times 3 \times 11 \times 9 = 594$.

Therefore, the fractions when reduced to a common denominator are $\frac{297}{594}$, $\frac{396}{594}$, $\frac{162}{594}$, and $\frac{132}{594}$.

2. Reduce $\frac{3}{8}$ of $\frac{7}{3}$, $\frac{6}{11}$ of $\frac{11}{7}$, and $\frac{40}{41}$ to equivalent fractions having a common denominator.

Ans. $\frac{2009}{2296}$, $\frac{1968}{2296}$, and $\frac{2240}{2296}$.

3. Reduce $\frac{7}{11}$, $\frac{13}{37}$ of $\frac{37}{17}$, and $\frac{41}{43}$ to equivalent fractions having a common denominator. Ans. $\frac{5117}{8041}$, $\frac{6149}{8041}$, and $\frac{7667}{8041}$.

4. Reduce $\frac{23}{29}$, $\frac{31}{47}$, $\frac{41}{43}$, and $\frac{47}{53}$, to equivalent fractions having a common denominator.

5. Reduce $\frac{73}{79}$, $\frac{83}{89}$, and $\frac{97}{101}$ to fractions having a common denominator.

6. Reduce $\frac{173}{977}$, and $\frac{1109}{1123}$ to fractions having a common denominator.

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22. To reduce fractions to their least common denominators, we have this

RULE.

Reduce the fractions to their simplest form. Then find the least common multiple of their denominators, (by rule under Art. 10, or rule under Art. 11,) which will be their least common

denominator. Divide this common denominator by the respective denominators of the given fractions, multiply the quotients by their respective numerators, and the products will be the new numerators.

The correctness of the above rule may be shown in the same way as was the preceding rule.

Examples.

1. Reduce $\frac{1}{2}$ of $\frac{3}{7}$ of $\frac{7}{12}$, $\frac{3}{20}$, and $\frac{7}{15}$, to equivalent fractions having the least common denominator.

These fractions when reduced to their simplest form become $\frac{1}{8}$, $\frac{3}{20}$, and $\frac{7}{15}$. The least common multiple of the denominators 8, 20, and 15, is 120=common denominator.

New numerator of first fraction is $\frac{120}{8}\times1=15$.
" " second fraction is $\frac{120}{20}\times3=18$.
" " third fraction is $\frac{120}{15}\times7=56$.

Hence the fractions, when reduced to their least common denominator, become $\frac{15}{120}$, $\frac{18}{120}$, and $\frac{56}{120}$.

2. Reduce $\frac{7}{8}$ of $\frac{4}{5}$, $4\frac{1}{2}$, and $\frac{3}{20}$ to equivalent fractions having a common denominator. Ans. $\frac{14}{20}$, $\frac{90}{20}$, and $\frac{3}{20}$.

3. Reduce $\frac{7}{8}$ of $\frac{3}{14}$ of $\frac{16}{12}$ of $\frac{24}{10}$, $\frac{11}{20}$, and $7\frac{1}{2}$ to fractions having the least common denominator. Ans. $\frac{12}{20}$, $\frac{11}{20}$, and $\frac{150}{20}$.

4. Reduce $\frac{11}{13}$ of $\frac{39}{44}$ of $\frac{4}{18}$ of $\frac{12}{7}$, $6\frac{1}{7}$, and $\frac{8}{28}$ to fractions having the least common denominator.

Ans. $\frac{2}{7}$, $\frac{43}{7}$, and $\frac{2}{7}$.

5. Reduce $\frac{3}{7}$ of $\frac{22}{28}$ of $\frac{20}{22}$ of $\frac{55}{49}$, $\frac{6}{22}$ of $\frac{3}{32}$, and $\frac{3}{33}$ of $\frac{22}{30}$ to fractions having the least common denominator.

6. Reduce $\frac{12}{7}$, $11\frac{4}{7}$, and $7\frac{5}{33}$ to fractions having the least common denominator.

7. Reduce $\frac{1}{2}$ of $\frac{4}{7}$ of $\frac{24}{30}$ of $\frac{30}{48}$, $3\frac{2}{28}$, and $10\frac{3}{56}$ to fractions having the least common denominator.

8. Reduce $\frac{17}{38}$, $\frac{13}{23}$ of $\frac{86}{95}$, and $\frac{37}{47}$ to fractions having the least common denominator.

9. Reduce $3\frac{1}{7}$, $5\frac{1}{5}$, $5\frac{1}{4}$, and $11\frac{9}{33}$ to fractions having the least common denominator.

ADDITION OF FRACTIONS.

23. Suppose we wish to add $\frac{3}{7}$ and $\frac{4}{5}$. We know that so long as these fractions are of different denominations they can not be added; we will therefore reduce them to a common denominator, we thus obtain $\frac{3}{7}=\frac{15}{35}$; $\frac{4}{5}=\frac{28}{35}$. Now, taking their sum we get $\frac{3}{7}+\frac{4}{5}=\frac{15}{35}+\frac{28}{35}=\dfrac{15+28}{35}=\frac{43}{35}=1\frac{8}{35}$.

Hence, to add fractions, we have this

RULE.

Reduce the fractions to a common denominator, and take the sum of the numerators, under which place the common denominator, and it will give the sum required.

Examples.

1. Add the fractions $\frac{1}{2}$ of $\frac{5}{7}$, $\frac{3}{11}$ of $\frac{11}{6}$, and $5\frac{1}{7}$.

These fractions reduced to their least common denominator are $\frac{5}{14}$, $\frac{7}{14}$, and $\frac{72}{14}$; and their sum is $\dfrac{5+7+72}{14}=\dfrac{84}{14}=6$.

2. Add the fractions $\frac{3}{10}$, $\frac{7}{5}$, $\frac{8}{15}$, and $\frac{1}{30}$. Ans. $\frac{68}{30}=2\frac{4}{15}$.

3. Add the fractions $\frac{5}{7}$ of $\frac{7}{8}$, $\frac{3}{16}$, and $4\frac{1}{2}$. Ans. $\frac{85}{16}=5\frac{5}{16}$.

4. Add the fractions $\frac{3}{4}$, $\frac{4}{5}$, $\frac{13}{17}$, and $\frac{2}{11}$. Ans. $\frac{9337}{3740}=2\frac{1857}{3740}$.

5. What is the sum of $\frac{7}{23}$, $\frac{13}{23}$, and $\frac{43}{92}$?

6. What is the sum of $\frac{240}{329}$, $\frac{722}{833}$, and $\frac{1}{3}$?

7. What is the sum of $\frac{49}{84}$, $\frac{23}{22}$, and $\frac{7}{9}$?

8. What is the sum of $\frac{1}{2}$ of $4\frac{1}{5}$, $\frac{3}{22}$ of $6\frac{1}{7}$, and $\frac{5}{8}$?

9. What is the sum of $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, $\frac{1}{5}$, $\frac{2}{6}$, and $\frac{1}{20}$?

10. What is the sum of $\frac{1}{2}$, $\frac{2}{3}$, $\frac{3}{4}$, $\frac{4}{5}$, $\frac{5}{6}$, and $\frac{9}{20}$?

11. What is the sum of $1\frac{1}{2}$, $2\frac{1}{3}$, $3\frac{1}{4}$, $4\frac{1}{5}$, and $5\frac{1}{6}$?

SUBTRACTION OF FRACTIONS.

24. To subtract one fraction from another we have this

RULE.

Reduce the fractions to a common denominator, and subtract the numerator of the subtrahend from that of the minuend; place the common denominator under the difference.

Examples.

1. From $\frac{3}{11}$ subtract $\frac{8}{31}$.

These fractions, when reduced to their least common denominator become $\frac{93}{341}$, $\frac{88}{341}$ $\therefore$ $\frac{3}{11}-\frac{8}{31}=\frac{93}{241}-\frac{88}{341}=\frac{5}{341}$.

2. From $\frac{17}{43}$ subtract $\frac{4}{29}$. Ans. $\frac{321}{1247}$.

3. From $2\frac{7}{8}$ subtract $\frac{19}{16}$. Ans. $\frac{27}{16}=1\frac{11}{16}$.

4. From $\frac{3}{8}$ of $\frac{4}{12}$ of $\frac{6}{7}$ subtract $\frac{1}{2}$ of $\frac{2}{19}$. Ans. $\frac{29}{532}$.

5. From $\frac{3}{17}$ subtract $\frac{1}{43}$. Ans. $\frac{112}{731}$.

6. From $\frac{49}{103}$ subtract $\frac{71}{1063}$. Ans. $\frac{44774}{109489}$.

7. From $\frac{13}{73}$ subtract $\frac{4}{62}$.

8. From $3\frac{1}{3}$ subtract $2\frac{3}{5}$.

9. From $\frac{1}{7}$ of $4\frac{1}{9}$ subtract $\frac{3}{8}$.

10. From $\frac{3}{34}$ of $\frac{2}{3}$ subtract $\frac{22}{36}$ of $\frac{2}{49}$.

11. From $\frac{6}{33}$ of $\frac{3}{8}$ subtract $\frac{2}{29}$.

12. From $\frac{67}{47}$ subtract $\frac{23}{37}$.

MULTIPLICATION OF FRACTIONS.

25. Multiply $\frac{3}{4}$ by $\frac{5}{7}$.

We have seen (under Art. 20) that $\frac{3}{4}$ multiplied by $\frac{5}{7}$ is the same as $\frac{3}{4}$ of $\frac{5}{7}$: Therefore we must use the same rule for multiplying fractions, as we would for reducing compound fractions.

Hence, to multiply together fractions, we have this

RULE.

Multiply all the numerators together for a new numerator, and all the denominators together for a new denominator: always observing to reject, or cancel, such factors as are common to both numerators and denominators.

Examples.

1. Multiply together the fractions $\frac{3}{11}$, $\frac{22}{21}$, $\frac{7}{9}$, and $\frac{1}{3}$.

Expressing the multiplication, we obtain $\frac{3}{11}\times\frac{22}{21}\times\frac{7}{9}\times\frac{1}{3}$.

Canceling the 3 and 7 of the numerators, against 21 of the denominators; also the 11 of the denominators against a part of the 22 of the numerators, we get $\frac{\cancel{3}}{\cancel{11}}\times\frac{\overset{2}{\cancel{22}}}{\cancel{21}}\times\frac{\cancel{7}}{9}\times\frac{1}{3}=\frac{2}{9\times3}=\frac{2}{27}$.

2. Multiply together the fractions $\frac{7}{11}$, $\frac{21}{35}$, $\frac{55}{42}$, and $\frac{4}{9}$.

Indicating the multiplication, we get $\frac{7}{11}\times\frac{21}{35}\times\frac{55}{42}\times\frac{4}{9}$. Canceling the 11 of the denominators, against a part of the 55 of the numerators; also the 7 of the numerators, against a part of 35 of the donominators, we obtain $\frac{\cancel{7}}{\cancel{11}}\times\frac{21}{\underset{5}{\cancel{35}}}\times\frac{\overset{5}{\cancel{55}}}{42}\times\frac{4}{9}$.

Again, canceling the 5 which is common to both numerators

and denominators; also the factor 7, which is common to 21 of the numerators and to 42 of the denominators, we get

$$\frac{\cancel{7}}{\cancel{11}}\times\frac{\overset{3}{\cancel{21}}}{\underset{\cancel{5}}{\cancel{35}}}\times\frac{\overset{\cancel{5}}{\cancel{55}}}{\underset{6}{\cancel{42}}}\times\frac{4}{9}.$$

Finally, canceling the 3 of the numerators, against a part of the 9 of the denominators; and the factor 2, which is common to the 4 of the numerators, and to 6 of the denominators, we obtain

$$\frac{\cancel{7}}{\cancel{11}}\times\frac{\overset{\cancel{3}}{\cancel{21}}}{\underset{\cancel{5}}{\cancel{35}}}\times\frac{\overset{\cancel{5}}{\cancel{55}}}{\underset{\underset{3}{\cancel{6}}}{\cancel{42}}}\times\frac{\overset{2}{\cancel{4}}}{\underset{3}{\cancel{9}}}=\frac{2}{3\times 3}=\frac{2}{9}.$$

NOTE.—A little practice will enable the student to perform these operations of canceling with great ease and rapidity. And since, as was remarked under Art. 20, it is immaterial which factors are first canceled, the simplicity of the work must depend much upon his skill or ingenuity.

3. Multiply together the fractions $\frac{3}{40}$, $\frac{7}{18}$, and $\frac{10}{14}$.

Ans. $\frac{1}{48}$.

4. Multiply together the fractions $\frac{17}{39}$, $\frac{14}{34}$, $\frac{26}{28}$, and $\frac{1}{11}$.

Ans. $\frac{1}{66}$.

5. Multiply together the fractions $\frac{3}{7}$, $3\frac{1}{3}$, $\frac{1}{8}$ of $\frac{16}{33}$, and $\frac{1}{11}$.

Ans. $\frac{20}{2541}$.

DIVISION OF FRACTIONS.

26. Divide $\frac{4}{7}$ by $\frac{5}{8}$.

We know that $\frac{4}{7}$ can be divided by 5, by multiplying the denominator by 5, (see Prop. II., Art. 15,) which gives $\frac{4}{7\times 5}$.

Now, since $\frac{5}{8}$ is but one-eighth of 5, it follows that $\frac{4}{7}$ divided by

$\frac{5}{8}$ must be eight times as great as $\frac{4}{7}$ divided by 5: ∴ $\frac{4}{7}$ divided by $\frac{5}{8}$ must be $\frac{4\times 8}{7\times 5}$. From this we see that $\frac{4}{7}$ has been multiplied by $\frac{5}{8}$ inverted.

Hence, to divide one fraction by another, we have this

RULE.

Reduce the fractions to their simplest form. Invert the divisor, and then proceed as in multiplication.

Examples.

1. Divide $\frac{43}{84}$ by $\frac{86}{21}$.

Inverting the divisor, and then multiplying, we obtain $\frac{43}{84} \times \frac{21}{86}$, which, by canceling, becomes $\frac{\cancel{43}}{\underset{4}{\cancel{84}}} \times \frac{\cancel{21}}{\underset{2}{\cancel{86}}} = \frac{1}{8}$.

2. Divide $\frac{107}{209}$ by $\frac{7}{418}$. Ans. $\frac{214}{7} = 30\frac{4}{7}$.
3. Divide $\frac{37}{49}$ by $\frac{98}{365}$. Ans, $\frac{13505}{4802} = 2\frac{3901}{4802}$.
4. Divide $\frac{15}{17}$ by $\frac{30}{3}$. Ans. $\frac{3}{34}$.
5. Divide $4\frac{1}{3}$ by $17\frac{1}{2}$. Ans. $\frac{26}{105}$.
6. Divide $\frac{1}{2}$ of $4\frac{1}{7}$ by $\frac{2}{3}$ of $\frac{5}{8}$. Ans. $\frac{174}{35} = 4\frac{34}{35}$.
7. Divide $\frac{3}{7}$ of $\frac{3}{8}$ of 7 by $\frac{13}{3}$ of 6.
8. Divide $\frac{14}{37}$ by $\frac{28}{35}$.
9. Divide $\frac{1}{2}$ of $\frac{3}{5}$ of $\frac{3}{4}$ by $\frac{3}{5}$ of $\frac{4}{3}$.
10. Divide $\frac{1}{20}$ of $\frac{1}{22}$ of $\frac{22}{7}$ by $\frac{3}{34}$.
11. Divide $\frac{1}{300}$ of $\frac{10}{33}$ by 3.

COMPLEX FRACTIONS.

27. Sometimes fractions occur in which the numerator or denominator, or both, are already fractional.

Thus, $\dfrac{2}{\frac{3}{7}}$, $\dfrac{\frac{2}{3}}{5}$, $\dfrac{\frac{3}{4}}{\frac{7}{8}}$, and $\dfrac{\frac{1}{9}}{\frac{1}{13}}$: such fractions are called *complex fractions.*

REDUCTION OF COMPLEX FRACTIONS.

28. Since the value of a fraction is the quotient arising from dividing the numerator by the denominator, it follows that $\dfrac{2}{\frac{3}{7}}$ is the same as $2 \div \frac{3}{7} = \frac{14}{3} = 4\frac{2}{3}$. Again, $\dfrac{\frac{3}{4}}{\frac{7}{8}} = \frac{3}{4} \div \frac{7}{8} = \frac{6}{7}$.

Hence, to reduce a complex fraction to a simple one, we have this

RULE.

Divide the numerator of the complex fraction by the denominator, according to rule under Art. 26.

Examples.

1. Reduce $\dfrac{4\frac{1}{2}}{3\frac{1}{3}}$ to a simple fraction.

Dividing $4\frac{1}{2} = \frac{9}{2}$ by $3\frac{1}{3} = \frac{10}{3}$, we get $\frac{27}{20} = 1\frac{7}{20}$.

2. Reduce $\dfrac{\frac{3}{7}}{\frac{6}{9}}$ to a simple fraction. Ans. $\frac{9}{14}$.

3. Reduce $\dfrac{3\frac{1}{7}}{\frac{4}{5}}$ to a simple fraction. Ans. $\frac{55}{14} = 3\frac{13}{14}$.

4. Reduce $\dfrac{6\frac{1}{7}}{3\frac{1}{4}}$ to a simple fraction. Ans. $\frac{172}{91} = 1\frac{81}{91}$.

5. Reduce $\dfrac{\frac{13}{4}}{4\frac{1}{2}}$ to a simple fraction. Ans. $\frac{13}{18}$.

6. Reduce $\frac{6\frac{9}{10}}{8}$ to a simple fraction. Ans. $\frac{69}{80}$.

7. Reduce $\frac{1\frac{3}{8}}{\frac{1}{2}\text{ of }\frac{1}{3}}$ to a simple fraction.

8. Reduce $\frac{\frac{3}{4}\text{ of }\frac{7}{9}}{\frac{2}{3}\text{ of }\frac{1}{5}}$ to a simple fraction.

9. Reduce $\frac{6+\frac{1}{8}}{7+\frac{1}{2}}$ to a simple fraction.

10. Reduce $\frac{10\frac{1}{23}}{11\frac{1}{32}}$ to a simple fraction.

REDUCTION OF FRACTIONS TO A GIVEN DENOMINATOR.

29. Suppose we wish to change $\frac{4}{5}$ to an equivalent fraction having 6 for its denominator.

It is obvious, that if we first multiply $\frac{4}{5}$ by 6, and then divide the product by 6, its value will not be altered. By this means we find that $\frac{4}{5}=\frac{\frac{4}{5}\times 6}{6}=\frac{\frac{24}{5}}{6}$.

Hence, to reduce a fraction to an equivalent fraction having a given denominator, we have this

RULE.

Multiply the fraction by the number which is to be the given denominator, (see rule under Art. 25,) under which place the given denominator, and it will be the fraction required.

Examples.

1. Reduce $\frac{3}{7}$ to an equivalent fraction, having 8 for its denominator.

In this example, we first multiply $\frac{3}{7}$ by 8, which gives $\frac{24}{7}$;

therefore, placing 8 under $\frac{24}{7}$ we get $\frac{\frac{24}{7}}{8}$ for the fraction required.

2. Reduce $\frac{3}{11}$ to an equivalent fraction having 12 for its denominator. Ans. $\frac{3\frac{3}{11}}{12}$.

3. Reduce $\frac{13}{10}$ to an equivalent fraction having 7 for its denominator. Ans. $\frac{9\frac{1}{10}}{7}$.

4. Reduce $\frac{1}{2}$, $\frac{3\frac{1}{3}}{10}$, $\frac{1}{4}$, and $\frac{1\frac{2}{3}}{10}$ to fractions having 12 for their common denominators. Ans. $\frac{6}{12}$, $\frac{4}{12}$, $\frac{3}{12}$, and $\frac{2}{12}$.

5. Reduce $\frac{1}{9}$, $\frac{1}{11}$, $\frac{1}{13}$ and $\frac{1}{17}$ to fractions having 100 for their common denominator. Ans. $\frac{11\frac{1}{9}}{100}$, $\frac{9\frac{1}{11}}{100}$, $\frac{7\frac{9}{13}}{100}$, and $\frac{5\frac{15}{17}}{100}$.

6. Reduce $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, $\frac{1}{5}$, and $\frac{1}{6}$ to fractions having 30 for their common denominator.

REDUCTION OF DENOMINATE FRACTIONS.

30. A *denominate fraction* is a fraction of a number of a particular denomination. Thus $\frac{7}{8}$ of a foot, $\frac{2}{3}$ of a yard, $\frac{1}{3}$ of a dollar, and $\frac{4}{5}$ of a shilling, are denominate fractions.

Reduction of denominate fractions is the changing of them from one denomination to another, without altering their values.

---

31. Suppose we wish to reduce $\frac{1}{360}$ of a pound sterling to an equivalent fraction of a farthing, we proceed as follows: Since there are 20 shillings in one pound, it follows that $\frac{1}{360}$ of a pound is the same as 20 times $\frac{1}{360}$ of a shilling, and this

is also the same as 12 times 20 times $\frac{1}{360}$ of a penny; which, in turn, is 4 times 12 times 20 times $\frac{1}{360}$ of a farthing. Hence, $\frac{1}{360}$ of a pound sterling is equivalent to $\frac{1}{360}$ of $\frac{20}{1}$ of $\frac{12}{1}$ of $\frac{4}{1}$ of a farthing.

Again, let us reduce $\frac{3}{5}$ of a farthing to an equivalent fraction of a pound sterling. In this case we must use the reciprocals of $\frac{20}{1}$, $\frac{12}{1}$, $\frac{4}{1}$, we thus find that $\frac{3}{5}$ of a farthing is equivalent to $\frac{3}{5}$ of $\frac{1}{4}$ of $\frac{1}{12}$ of $\frac{1}{20}$ of a pound sterling.

Hence, to reduce fractions of one denominate value to equivalent fractions of other denominate values, we have this

RULE.

I. *When the given fraction is to be reduced to a higher denomination, multiply it by a compound fraction whose terms are the reciprocals of the successive denominate values, included between the denomination of the given fraction, and the one to which it is to be reduced.*

II. *When the given fraction is to be reduced to a lower denomination, then multiply it by a compound fraction, whose terms have units for their denominators, and for numerators the successive denominate values included between the denomination of the given fraction, and the one to which it is to be reduced.*

Examples.

1. Reduce $\frac{3}{8}$ of an inch to the fraction of a mile.

In this example, the different denominate values between an inch and a mile are 12 inches, $16\frac{1}{2}=\frac{33}{2}$ feet, 40 rods, and 8 furlongs; $\therefore$ our compound fraction is $\frac{1}{12}$ of $\frac{2}{33}$ of $\frac{1}{40}$ of $\frac{1}{8}$, which multiplied by the given fraction produces $\frac{3}{8}$ of $\frac{1}{12}$ of $\frac{2}{33}$ of $\frac{1}{40}$ of $\frac{1}{8}$: canceling the 3 and 2 of the numerators against

a part of the 12 of the denominator we get $\frac{\cancel{3}}{8}\times\frac{1}{\underset{2}{\cancel{12}}}\times\frac{\cancel{2}}{33}\times\frac{1}{40}\times\frac{1}{8}=\frac{1}{168960}$. Therefore, $\frac{3}{8}$ of an inch is equivalent to $\frac{1}{168960}$ of a mile.

2. Reduce $\frac{3}{11520}$ of a solar day to an equivalent fraction of a second.

In this example the successive denominate values between a solar day and a second are 24 hours, 60 minutes, and 60 seconds; therefore our compound fraction is $\frac{24}{1}$ of $\frac{60}{1}$ of $\frac{60}{1}$, which, multiplied by the given fraction, becomes $\frac{3}{11520}$ of $\frac{24}{1}$ of $\frac{60}{1}$ of $\frac{60}{1}$; this becomes, after canceling like factors, $\frac{45}{2}$ of a second.

3. Reduce $\frac{144}{360}$ of a yard to the fraction of a mile.

Ans. $\frac{1}{4400}$.

4. Reduce $\frac{27}{84}$ of a gill to the fraction of a gallon.

Ans. $\frac{9}{896}$.

5. Reduce $\frac{230}{345}$ of a pound to the fraction of a ton.

Ans. $\frac{1}{3360}$.

6. Reduce $\frac{1}{3}$ of a mile to the fraction of a foot.

Ans. 1760 feet.

7. Reduce $\frac{1}{2}$ of $\frac{1}{3}$ of $\frac{2}{7}$ of a yard to the fraction of a mile.

Ans. $\frac{1}{36960}$.

8. Reduce $\frac{1}{6}$ of $\frac{1}{8}$ of $\frac{24}{28}$ of a gallon to the fraction of a gill.

Ans. $\frac{4}{7}$.

9. Reduce $\frac{3}{8}$ of $\frac{4}{9}$ of a hogshead of wine to the fraction of a gill.

10. Reduce $\frac{1}{3}$ of $\frac{3}{7}$ of $4\frac{1}{2}$ yards to the fraction of an inch.

11. Reduce $\frac{1}{7}$ of $\frac{3}{43}$ of a farthing to the fraction of a shilling.

12. Reduce $\frac{7}{80}$ of an ounce to the fraction of a pound avoirdupois.

32. To find what fractional part one quantity is of an another of the same kind, but of different denominations.

Suppose we wish to know what part of 1 yard 2 feet 3 inches is; we reduce 1 yard to inches, which gives 1 yard =36 inches; we also reduce 2 feet 3 inches to inches, which gives 2 feet 3 inches=27 inches. Now it is obvious that 2 feet 3 inches is the same part of one yard that 27 is of 36, which is $\frac{27}{36}=\frac{3}{4}$.

Hence, we deduce this

RULE.

Reduce the given quantities to the lowest denomination mentioned in either, then divide the number which is to become the fractional part, by the other number.

Examples.

1. What part of £3 4*s.* 1*d.* is 2*s.* 6*d.*?

In this example the quantities when reduced become £3. 4*s.* 1*d.*=769*d.*; and 2*s.* 6*d.*=30*d.*; therefore $\frac{30}{769}$ is the fractional part which 2*s.* 6*d.* is of £3. 4*s.* 1*d.*

2. What part of 3 miles 40 rods is 27 feet 9 inches?

Ans. $\frac{37}{22000}$.

3. What part of a day is 17 minutes 4 seconds?

Ans. $\frac{8}{675}$.

4. What part of $700 is $5.30? Ans. $\frac{53}{7000}$.

5. What fractional part of 2 hogsheads is 3 pints?

Ans. $\frac{1}{336}$.

6. What part of $3 is $2\frac{1}{2}$ cents? Ans. $\frac{1}{120}$.

7. What part of 10 shillings 8 pence is 3 shillings 1 penny?

Ans. $\frac{37}{128}$.

8. What part of 100 acres is 63 acres 2 roods and 7 rods of land? Ans. $\frac{10167}{16000}$.

5

33. To reduce a fraction of any given denomination to whole denominate numbers.

Suppose we wish to know the value of $\frac{3}{8}$ of a yard; we know that $\frac{3}{8}$ of a yard equals $\frac{3}{8}$ of $\frac{4}{1}$ of a quarter $=\frac{3}{2}$ of a quarter $=1$ quarter $+\frac{1}{2}$ of a quarter.

Again, $\frac{1}{2}$ of a quarter equals $\frac{1}{2}$ of $\frac{4}{1}$ of a nail $=2$ nails. Therefore $\frac{3}{8}$ of a yard equals 1 quarter 2 nails. Hence, we deduce this

RULE.

Multiply the numerator by the units in the next inferior denominate value, and divide the product by the denominator; multiply the remainder, if any, by the next lower denominate value, and again divide the product by the denominator; continue this process until there is no remainder, or until we reach the lowest denominate value. The successive quotients will form the successive denominate values.

Examples.

1. What is the value of $\frac{3}{15}$ of an hour?

In this example, $\frac{3}{15}$ of an hour $=\frac{3}{15}$ of $\frac{60}{1}$ of a minute $=12$ minutes.

2. What is the value of $\frac{3}{7}$ of 1 yard?

Ans. 1 quarter $2\frac{6}{7}$ nails.

3. What is the value of $\frac{1}{2}$ of $\frac{3}{8}$ of one mile?

Ans. 1 furlong 20 rods.

4. What is the value of $\frac{3}{7}$ of $\frac{5}{6}$ of 1 cwt.

Ans. 1 quarter 12 pounds.

5. What is the value of $\frac{1}{6}$ of 14 miles 6 furlongs?

Ans. 2 miles 3 furlongs 26 rods 11 feet.

6. What is the value of $\frac{1}{4}$ of $\frac{4}{5}$ of 2 days of 24 hours each?

Ans. 9 hours 36 minutes.

7. What is the value of $\frac{1}{2}$ of $\frac{7}{8}$ of $\frac{3}{14}$ of an hour?

Ans. 5 minutes $37\frac{1}{2}$ seconds.

CHAPTER III.

DECIMAL FRACTIONS.

34. A *decimal fraction* is that particular form of a vulgar fraction whose denominator consists of a unit followed by one or more ciphers.

Thus, $\frac{6}{10}$, $\frac{37}{100}$, $\frac{4}{1000}$, and $\frac{3634}{10000}$, are decimal fractions.

In practice, the denominators of decimal fractions are not written; but they are always understood.

Thus, instead of $\frac{3}{10}$, $\frac{7}{100}$, $\frac{37}{1000}$, and $\frac{1}{10000}$, we write 0.3; 0.07; 0.037; and 0.0001.

The first figure on the right of the period, or *decimal point*, is said to be in the place of tenths, the second figure is said to be in the place of hundredths, the third in the place of thousandths, and so on, decreasing from the left towards the right, in a tenfold ratio, the same as in whole numbers. The following table will exhibit this more clearly.

NUMERATION TABLE OF WHOLE NUMBERS AND DECIMALS.*

&c. &c.	Tens of Billions.	Billions.	Hundreds of Millions.	Tens of Millions.	Millions.	Hundreds of Thousands.	Tens of Thousands.	Thousands.	Hundreds.	Tens.	UNITS.	Decimal Point.	Tenths.	Hundredths.	Thousandths.	Ten Thousandths.	Hundred Thousandths.	Millionths.	Ten Millionths.	Hundred Millionths.	Billionths.	Ten Billionths.	&c. &c.
	3	3	3	3	3	3	3	3	3	3	3	.	3	3	3	3	3	3	3	3	3	3	

Ascending. Descending.

* This table is in accordance with the French method of numbering, where each period of three figures changes its denominate value.

Examples.

1. Write 7 tenths; 365 thousandths; 75 millionths.
Ans. 0.7; 0.365; 0.000075.

2. Write 37 hundredths; 5 tenths; 3781 ten millionths.
Ans. 0.37; 0.5; 0.0003781.

3. Write 43 hundredths; 3456 ten thousanths.
Ans. 0.43; 0.3456.

4. Write 13 billionths; 3 ten billionths.
Ans. 0.000000013; 0.0000000003.

---

35. Since decimals, like whole numbers, decrease from the left towards the right in a tenfold ratio, they may be connected together by means of the decimal point, and then operated upon by precisely the same rules as for whole numbers, provided we are careful to keep the decimal point always in the right place.

Annexing a cipher to a decimal does not change its value. Thus, 0.3=0.30=0.300=&c. But prefixing a cipher, is the same as removing the decimal figures one place further to the right, and therefore, each cipher thus prefixed reduces the value in a tenfold ratio.

Thus, 0.3 is ten times 0.03, or a hundred times 0.003.

ADDITION OF DECIMALS.

36. From what has been said under Art. 35, we deduce the following

RULE.

Place the numbers so that the decimal points shall be directly over each other, and then add as in whole numbers.

Examples.

1. Find the sum of 47.3; 37.672; 1.789101; 88.9134; and 0.0037.

OPERATION.

```
     47.3
     37.672
      1.789101
     88.9134
      0.0037
Ans. 175.678201
```

2. What is the sum of 0.67; 0.0371; 47.5; 1100.0001; 29.0037; 1.000095; and 33.033? Ans. 1211.243905.

3. What is the sum of 1.8; 40.06; 120.365; 1100.0001; 47.003; 31.11101; and 3.0001? Ans. 1343.33921.

4. What is the sum of 13.29; 14.2835; 111.117; 4.006; 67.88864; and 496.446? Ans. 707.03114.

5. What is the sum of 37.345; 8.26; 7.534; 19.0005; 10.94; and 103.729? Ans. 186.8085.

6. What is the sum of 0.90058; 7.634; 3.007956; 1.1? Ans. 12.642536.

7. What is the sum of 47.635; 3.13; 4.5787; 0.003001; 0.40005; and 4112.3789? Ans. 4168.125651.

8. What is the sum of 17.154; 32.004501; 6.4; 49.345; and 1.0005? Ans. 105.904001.

9. What is the sum of 4.996; 38.37; 421.633; 5.65; and 4.29? Ans. 474.939.

10. What is the sum of 57.41; 365.0001; and 1.101?

11. What is the sum of 2.4999; 47.121212; 0.1; and 411.001?

12. What is the sum of 433.9; 777.5; 67.06; and 35.88?

SUBTRACTION OF DECIMALS.

37. From what has been said under Art. 35, we infer the following

RULE.

Place the smaller number under the larger, so that the decimal point of the one is directly under that of the other. Then proceed as in subtraction of whole numbers.

Examples.

1. From 213.5734 subtract 87.657237.

$$\begin{array}{rr} & 213.5734 \\ & 87.657237 \\ \hline \text{Ans.} & 125.916163 \end{array}$$

2. From 385.76943 subtract 72.57. Ans. 313.19943.
3. From 0.975 subtract 0.483764. Ans. 0.491236.
4. From 0.5 subtract 0.0003. Ans. 0.4997.
5. From 96.5 subtract 0.000783. Ans. 96.499217.
6. From 23.005 subtract 13.000378. Ans. 10.004622.
7. From 110.001001 subtract 11.010002.
 Ans. 98.990999.

MULTIPLICATION OF DECIMALS.

38. Let us multiply 0.47 by 0.6. If we put these decimals in the form of vulgar fractions they become $\frac{47}{100}$ and $\frac{6}{10}$; these multiplied, by rule under Art. 25, give $\frac{47}{100} \times \frac{6}{10} = \frac{282}{1000}$. Now it is obvious that there will be, in all cases, as many ciphers in the denominator of the product as there are in both of the factors added together.

Hence, the following

RULE.

Multiply the two factors after the same manner as in whole numbers, then point off from the right of the product as many figures for decimals as there are decimal places in both the factors. If there are not so many places of figures, supply the deficiency by prefixing ciphers.

Examples.

1. Multiply 3.753 by 1.656.

OPERATION.

```
        3.753
        1.656
       ------
        22518
       18765
      22518
      3753
     --------
Ans. 6.214968
```

2. What is the product of 0.005 into 0.017 ?

Ans. 0.000085.

3. What is the product of 0.376 into 0.0076894 ?

Ans. 0.0028912144.

4. What is the product of 0.576 into 0.3854 ?

Ans. 0.2219904.

5. What is the product of 0.43 into 0.65 ?

Ans. 0.2795.

6. What is the product of 3.9765 into 4.378 ?

Ans. 17.409117.

7. What is the product of 415.314 into 7.3004 ?

Ans. 3031.9583256.

8. What is the product of 7.42 into 11.1415 ?

Ans. 82.66993.

ABRIDGED MULTIPLICATION OF DECIMALS.

39. If we multiply 0.37894 by 0.67452, by the last rule, our work will be as follows:

OPERATION.

$$
\begin{array}{r|l}
0.3 & 7894 \\
0.6 & 7452 \\
\hline
7 & 5788 \\
189 & 470 \\
1515 & 76 \\
26525 & 8 \\
227364 & \\
\hline
\text{Ans. } 0.255602 & 6088
\end{array}
$$

Here it is plain that the figures on the right of the vertical line can not be depended upon as accurate, since they are beyond the sixth place of decimals, whilst the factors are supposed to be true to only five decimals: so that if we wish to be accurate we must reject all the decimals on the right of the vertical line. By the rule below, we may perform the multiplication so as to exclude all that part of the work on the right of our vertical line, thereby shortening the work, and still obtaining the product with the same degree of accuracy as by the usual rule.

Our rule for contracting the work of multiplying decimals is as follows:

RULE.

I. *Multiply the multiplicand by the left-hand figure of the multiplier.*

II. *Multiply the multiplicand, deprived of its right-hand figure, by the second figure of the multiplier, counting from the left.*

III. *Multiply the multiplicand, deprived of its two right-*

hand figures, by the third figure of the multiplier, counting from the left.

Continue this process until all the figures of the multiplier have been used. Observe to place the successive products so that their right-hand figures shall be directly under each other.

NOTE.—In omitting successively the different figures on the right of the multiplicand, we must so far use them as to determine what there would be to carry into the next column.

Examples.

1. Multiply 0.37894 by 0.67452.

OPERATION.

	0.37894
	0.67452
	227364
	26526
	1516
	189
	7
Ans.	0.255602

EXPLANATION.

First. We multiply the multiplicand 0.37894 by 6, the left-hand figure of the multiplier, which gives the first partial product 227364.

Secondly. We multiply 0.3789, which is the multiplicand deprived of its right-hand figure, by 7, the second figure of the multiplier, observing to carry 3, since the figure cut off multiplied by 7 gives 28, which is nearer 30 than 20: we thus obtain 26526 for the second partial product.

Thirdly. Multiplying 0.378 by 4, observing to carry 4, we obtain 1516 for the third partial product.

Fourthly. Multiplying 0.37 by 5, observing to carry 4, we obtain 189 for the fourth partial product.

Fifthly. Multiplying 0.3 by 2, observing to carry 1, we get 7 for the fifth partial product.

2. Multiply 0.3785 by 0.4673.

OPERATION.

```
      0.3785
      0.4673
      ------
       15140
        2271
         265
          11
      ------
Ans. 0.17687
```

3. What is the product of 0.00524486 by 0.99993682?

OPERATION.

```
      0.00524486
      0.99993682
      ----------
         4720374
          472037
           47204
            4720
             157
              31
               4
     -----------
Ans. 0.005244527
```

4. Multiply 108.2808251671 by 1.9614591767.

Ans. 212.3884181846.

5. Multiply 0.009416517988 by 0.999936883996.

Ans. 0.0094159236548.

6. Multiply 0.0000375229 by 0.0000275177.

7. Multiply 0.999936883996 by 0.999955663612.

8. Multiply 0.587401052 by 0.018468950.

9. Multiply 91.6264232009 by 0.0172021234.

10. Multiply 212.3880258928 into itself.

DIVISION OF DECIMALS.

40. In multiplication we have seen that there are as many decimal places in the product as there are in both the factors; and since division is the reverse of multiplication, it follows that the number of decimal places in the quotient must equal the excess of those in the dividend, above those of the divisor: Hence, to divide one decimal expression by another we have this

RULE.

Divide as in whole numbers; and point off as many places from the right of the quotient, for decimals, as the decimal places in the dividend exceed those of the divisor. If there are not so many figures in the quotient as this excess, supply the deficiency by prefixing ciphers.

Examples.

1. Divide 3.475 by 4.789.

OPERATION.

```
4.789)3.475000(0.725
       33523
       -----
        12270
         9578
         -----
         26920
         23945
         -----
          2975
```

In this example the number of decimal places in the dividend, including the ciphers which were annexed, is 6, whilst the number of places in the divisor is 3; therefore we make 3 places of decimals in the quotient. We might continue to annex ciphers to the remainder, and thus obtain additional decimal figures.

2. What is the quotient of 78.56453 divided by 4.78?

Ans. 16.436.

3. What is the quotient of 1561.275 divided by 24.3 ?

Ans. 64.25.

4. What is the quotient of 0.264 divided by 0.2 ?

Ans. 1.32.

5. What is the quotient of 3.52275 divided by 3.355 ?

6. What is the quotient of 1001.25 divided by 2.25 ?

ABRIDGED DIVISION OF DECIMALS.

42. If we divide 0.30679006 by 0.27610603, by the last rule, our work will be as follows:

OPERATION.

```
0.27610603)0.30679006(1.1111313
            27610603
             3068403|0
             2761060|3
              307342|70
              276106|03
               31236|670
               27610|603
                3626|0670
                2761|0603
                 865|00670
                 828|31809
                  36|688610
                  27|610603
                   9|0780070
                   8|2831809
                    |7948261
```

By simply inspecting the above work, it is obvious that all that part of the work which is on the right of the vertical line, can in no way affect the accuracy of our quotient figures.

By the following rule we may perform the work of division so as to exclude all that part of the work on the right of the vertical line, thereby shortening the work, and still obtaining as accurate a result as by the last rule.

To contract the work in division of decimals, we have this

RULE.

Proceed as in the last rule, until we reach that point of the work where it would be necessary to annex ciphers to the remainder. Then, instead of annexing a cipher to the remainder, omit the right-hand figure of the divisor, and we shall obtain the next figure of the quotient; and thus continue, at each successive figure of the quotient to omit the right-hand figure of the divisor until there is but one figure in the remainder.

Examples.

1. What is the quotient of 365.424907 divided by 0.263803?

```
0.263803)365.424907(1385.21892
         263803
         -------
         1016219
          791409
          -------
          2248100
          2110424
          -------
           1376767
           1319015
           -------
             57752
             52761
             -----
              4991
              2638
              ----
              2353
              2110
              ----
               243
               237
               ---
                 6
                 5
                 -
                 1
```

2. What is the quotient of 0.123456 divided by 1.912478?

OPERATION.

```
1.912478)0.123456(0.064552
         114748
           8708
           7650
           1058
            956
            102
             96
              6
              4
              2
```

3. What is the quotient of 0.5260000 divided by 0.5260202?
Ans. 0.9999616.

4. What is the quotient of 7.45678 divided by 4.56789?
Ans. 1.63244.

5. What is the quotient of 7.632038 divided by 3.716048?

6. What is the quotient of 2 divided by 15.314865?

7. What is the quotient of 0.926954 divided by 0.3547898?

8. What is the quotient of 13.75892 divided by 6.76897?

---

42. To change a vulgar fraction into an equivalent decimal fraction.

It is obvious that the rule under Art. 33, will apply to this case, by considering all the denominate values as decreasing regularly in a tenfold ratio. Hence, this

RULE.

Annex a cipher to the numerator, and then divide by the denominator; to the remainder annex another cipher, and again divide by the denominator: and so continue until there is no

remainder, or until we have obtained as many decimal figures as may be desired. The successive quotients will be the successive decimal figures required.

Examples.

1. What decimal fraction is equivalent to $\frac{1}{16}$?

$$\begin{array}{l} 16)100(0.0625 \text{ Ans.} \\ \underline{\;\;96} \\ \;\;\;\;40 \\ \;\;\;\;\underline{32} \\ \;\;\;\;\;\;80 \\ \;\;\;\;\;\;\underline{80} \\ \;\;\;\;\;\;\;0 \end{array}$$

2. What decimal is equivalent to $\frac{1}{18}$?

Ans. 0.05555, &c.

3. What decimal is equivalent to $\frac{1}{20}$? Ans. 0.05.

4. What decimal is equivalent to $\frac{1}{25}$? Ans. 0.04.

5. What decimal is equivalent to $\frac{1}{3}$? Ans. 0.3333, &c.

---

43. It will often happen, as in examples 2 and 5, under Art. 42, page 58, that the process will never terminate, in which case there is no decimal value which is accurately equal to the vulgar fraction.

Since we constantly multiplied the remainders by 10, it follows that whenever the denominator of the vulgar fraction contains no prime factors different from those which compose 10, viz. 2 and 5, then the decimal value will *terminate*. But in all other cases, the decimal expression must consist of an infinite number of figures.

Hence, to determine whether a given vulgar fraction can be accurately expressed in decimals, we have this

RULE.

Decompose the denominator of the vulgar fraction into its prime factors, (by rule under Art. 6,) then, if there are no prime factors different from 2 and 5, the vulgar fraction can be accurately expressed by decimals; but if it contain different factors, it can not be accurately expressed in decimals.

Examples.

1. Can the vulgar fraction $\frac{3}{386}$ be accurately expressed in decimals?

In this example we find that $386=2\times 193$; so that the denominator contains the prime factor 193, which is different from 2 or 5; consequently, $\frac{3}{386}$ can not be accurately expressed in decimals.

2. Can the vulgar fraction $\frac{17}{812}$ be accurately expressed by decimals? Ans. It can not.

3. Can the vulgar fractions, having for denominators 640, be expressed in decimals accurately? Ans. They can.

---

44. When a vulgar fraction can be accurately expressed in decimals, we may determine the number of decimal places by the following

RULE.

Decompose the denominator into its prime factors, (by rule under Art. 6,) which factors can not differ from 2 and 5, (by rule under Art. 43.) The highest exponent of 2 or 5 will be the number of decimal places sought.

Examples.

1. How many places of decimals will be required to express $\frac{1}{40}$?

In this example we find $40=2^3\times 5$, where the highest exponent is 3; therefore the number of decimal places is 3.

2. How many places of decimals will be required to express $\frac{7}{125}$? Ans. 3.

3. How many places of decimals will be required to express $\frac{463}{3125}$? Ans. 5.

4. How many places of decimals will be required to express $\frac{13}{16}$?

5. How many places of decimals will be required to express $\frac{37}{160}$?

6. How many places of decimals will be required to express $\frac{43}{1600}$?

---

45. When the decimal figures obtained by converting a vulgar fraction into decimals do not terminate, they must recur in periods, whose number of terms can not exceed the number of units in the denominator, less one. For all the different remainders which occur must be less than the denominator; and therefore their number can not exceed the denominator, less one; and whenever we obtain a remainder like one that has previously occurred, then the decimal figures will begin to repeat. Decimals which recur in this way are called *repetends.*

When the period begins with the first decimal figure it is called a *simple repetend.* But when other decimal figures occur before the period commences, it is called a *compound repetend.*

A repetend is distinguished from ordinary decimals by a period or dot placed over the first and last figure of the circulating period.

46. The following vulgar fractions give simple repetends.

$\frac{1}{3}=0.\dot{3}.$

$\frac{1}{7}=0.\dot{1}4285\dot{7}.$

$\frac{1}{9}=0.\dot{1}.$

$\frac{1}{11}=0.\dot{0}\dot{9}.$

$\frac{1}{13}=0.\dot{0}7692\dot{3}.$

$\frac{1}{17}=0.\dot{0}58823529411764\dot{7}.$

$\frac{1}{19}=0.\dot{0}5263157894736842\dot{1}.$

$\frac{1}{21}=0.\dot{0}4761\dot{9}.$

$\frac{1}{23}=0.\dot{0}43478260869565217391\dot{3}.$

---

47. The following ones give compound repetends.

$\frac{1}{6}=0.1\dot{6}.$

$\frac{1}{12}=0.08\dot{3}.$

$\frac{1}{14}=0.0\dot{7}1428\dot{5}.$

$\frac{1}{15}=0.0\dot{6}.$

$\frac{1}{18}=0.0\dot{5}.$

$\frac{1}{22}=0.0\dot{4}\dot{5}.$

$\frac{1}{24}=0.041\dot{6}.$

---

48. Whenever the prime factors of the denominator of a vulgar fraction contain neither of the factors 2 and 5, the repetend will be simple. But when they contain one or both of the factors 2 and 5, together with other factors, then the repetend will be compound.

49. Those simple repetends, which have as many terms, less one, as there are units in the denominator, we shall call *perfect repetends.* The following are some of the perfect repetends.

$\frac{1}{7}=0.\dot{1}4285\dot{7}.$

$\frac{1}{17}=0.\dot{0}58823529411764\dot{7}.$

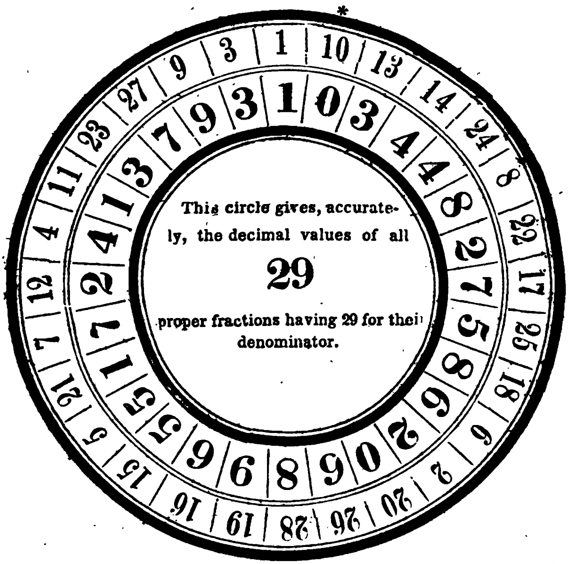
$\frac{1}{19}=0.\dot{0}5263157894736842\dot{1}.$

$\frac{1}{23}=0.\dot{0}43478260869565217391\dot{3}.$

$\frac{1}{29}=0.\dot{0}34482758620689655172413793\dot{1}.$

---

50. Perfect repetends possess some very remarkable properties, which we will explain by means of the following figure.



In this figure, the inner circle of figures, commencing at the O, directly under the *asterisk*, and counting towards the right-hand, is the circulating period of $\frac{1}{29}$.

The outer circle of figures, commencing at the same point and counting in the same direction, are the successive remainders, which will occur in the operation of decimating $\frac{1}{29}$, (by rule under Art. 42, page 58.)

In this circle of remainders, all the numbers, from 1 to 28 inclusive, occur; but not in numerical order.

By inspecting the above figure, we discover the following properties, which are common to all *perfect repetends.*

I. The sum of any two diametrically opposite figures, of the circle of decimals, will be 9.

II. The sum of any two diametrically opposite terms, in the circle of remainders, will make the denominator 29.

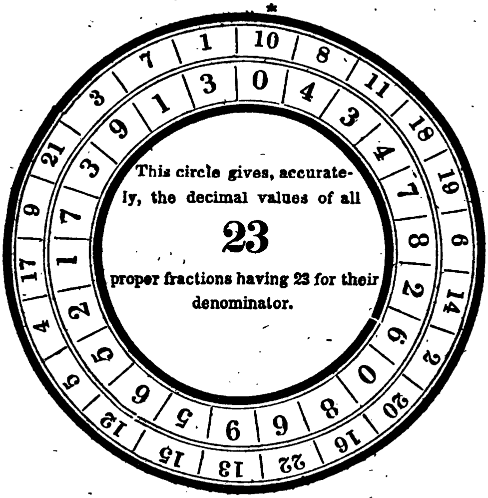
III. If we subtract the right-hand figure of the denominator from 10, *and multiply the remainder by any decimal figure of the inner circle, the right-hand figure of the product will be the same as the right-hand figure of the corresponding remainder of the outer circle.*

IV. Commencing the circle of decimals at any point, and counting completely round, it will be the perfect repetend of the vulgar fraction, whose denominator is the same as in the first case, but whose numerator is the remainder in the outer circle, standing one place to the left.

From the IV. property it follows, that this same circle of decimals expresses the decimal value of all proper vulgar fractions whose denominators are 29.

The following figure, formed from the *perfect repetend* of

the value of $\frac{1}{23}$, possesses similar properties to those just explained.



Similar circles may be formed for all *perfect repetends.*

---

51. If we take the *perfect repetend* arising from $\frac{1}{7}$, we find

$$\frac{1}{7}=0.\dot{1}4285\dot{7}$$
$$\frac{2}{7}=0.\dot{2}8571\dot{4}$$
$$\frac{3}{7}=0.\dot{4}2857\dot{1}$$
$$\frac{4}{7}=0.\dot{5}7142\dot{8}$$
$$\frac{5}{7}=0.\dot{7}1428\dot{5}$$
$$\frac{6}{7}=0.\dot{8}5714\dot{2}$$

Taking the *perfect repetend* arising from $\frac{1}{17}$ we find

$$\frac{1}{17}=0.\dot{0}58823529411764\dot{7}$$
$$\frac{2}{17}=0.\dot{1}17647058823529\dot{4}$$
$$\frac{3}{17}=0.\dot{1}76470588235294\dot{1}$$
$$\frac{4}{17}=0.\dot{2}35294117647058\dot{8}$$
$$\frac{5}{17}=0.\dot{2}94117647058823\dot{5}$$
$$\frac{6}{17}=0.\dot{3}52941176470588\dot{2}$$
$$\frac{7}{17}=0.\dot{4}11764705882352\dot{9}$$
$$\frac{8}{17}=0.\dot{4}70588235294117\dot{6}$$
$$\frac{9}{17}=0.\dot{5}29411764705882\dot{3}$$
$$\frac{10}{17}=0.\dot{5}88235294117647\dot{0}$$
$$\frac{11}{17}=0.\dot{6}47058823529411\dot{7}$$
$$\frac{12}{17}=0.\dot{7}05882352941176\dot{4}$$
$$\frac{13}{17}=0.\dot{7}64705882352941\dot{1}$$
$$\frac{14}{17}=0.\dot{8}23529411764705\dot{8}$$
$$\frac{15}{17}=0.\dot{8}82352941176470\dot{5}$$
$$\frac{16}{17}=0.\dot{9}41176470588235\dot{2}$$

NOTE.—The I., II., and III. of the above properties of *perfect repetends* have never been noticed by any author. I have given an algebraic demonstration of these properties, as well as several others, which was published in the American Journal of Science, Vol. 40, No. 81.

52. When many figures in the decimal are required, we may proceed as follows :

Required the decimal value of $\frac{1}{29}$.

Operating by the rule under Art. 42, page 58, we get

OPERATION.

$$\begin{array}{l}29)100(0.03448\\ \quad\;\; 87\\ \quad\;\; \overline{130}\\ \quad\;\; 116\\ \quad\;\;\; \overline{140}\\ \quad\;\;\; 116\\ \quad\;\;\;\; \overline{240}\\ \quad\;\;\;\; 232\\ \quad\;\;\;\;\; \overline{8}\end{array}$$

We have continued this process until we have found a remainder consisting of but one figure; placing this remainder, when divided by 29, at the right of the quotient, agreeably to the usual rules of division, we get,

I. $\frac{1}{29}=0.03448\frac{8}{29}$. Multiplying this by 8 we get $\frac{8}{29}=0.27586\frac{6}{29}$. Substituting this value of $\frac{8}{29}$ in I., we get,

II. $\frac{1}{29}=0.0344827586\frac{6}{29}$; this, multiplied by 6, gives $\frac{6}{29}=0.2068965517\frac{7}{29}$; which substituted in II. gives,

III. $\frac{1}{29}=0.03448275862068965517\frac{7}{29}$. Again, multiplying this by 7, we get $\frac{7}{29}=0.24137931034482758620\frac{19}{29}$. Substituting this in III. we get,

IV. $\frac{1}{29}=0.03448275862068965517241379310344827586$-$20\frac{19}{29}$.

We shall gain nothing by continuing this process farther, since (by Art. 45) we know that the decimal figures can not extend beyond 28 places without repeating; in this case the number of places in the repetend is exactly 28 : it is therefore a *perfect repetend*.

53. There is another way of decimating, which is as follows:

Decimate $\frac{1}{97}$.

According to rule under Art. 42, we find

$$\begin{array}{r} 97)100(0.01 \\ \underline{97} \\ 3 \end{array}$$

To continue the process we must add ciphers to this remainder, in the same way as we did to the numerator, 1. Now the remainder being 3 times as large as the first numerator, it follows that the next two decimal figures must be 3 times the two just obtained, that is $3\times01=03$; and for a similar reason we must multiply 03 by 3 to obtain the next two figures, and so on. Proceeding in this way we find

$$\begin{array}{l} \frac{1}{97}=0.0103092781 \\ \qquad\qquad\qquad\quad 243 \\ \qquad\qquad\qquad\qquad 729 \\ \qquad\qquad\qquad\qquad\; 2187 \\ \qquad\qquad\qquad\qquad\quad 6561 \\ \qquad\qquad\qquad\qquad\quad\; \&c. \\ \hline \frac{1}{97}=0.010309278350515261 \;\&c. \end{array}$$

Decimating $\frac{1}{4}$ by the above plan we get

$$\begin{array}{l} \frac{1}{4}=0.248 \\ \qquad\;\; 16 \\ \qquad\quad 32 \\ \qquad\quad\; 64 \\ \qquad\quad\; 128 \\ \qquad\qquad 256 \\ \qquad\qquad \&c. \\ \hline \end{array}$$

$\frac{1}{4}=0.24999$ &c., which will constantly approximate towards 0.25; hence $\frac{1}{4}=0.25$.

Decimating $\frac{1}{8}$ by this method we get

$$\begin{array}{l} \frac{1}{8}=0.1248 \\ \quad\quad\quad 16 \\ \quad\quad\quad\; 32 \\ \quad\quad\quad\quad 64 \\ \quad\quad\quad\quad 128 \\ \quad\quad\quad\quad\; 256 \\ \quad\quad\quad\quad\; \&c. \\ \hline \quad 0.1249999\ \&c. \end{array}$$

0.1249999 &c. which will constantly approximate towards 0.125.

---

54. To change a decimal fraction into an equivalent vulgar fraction.

CASE I.

When the number of places is *finite*, we can, from the definition of decimal fractions, Art. 34, deduce this

RULE.

Make the given decimal the numerator of the vulgar fraction, and for its denominator write 1, *with as many ciphers annexed as there are decimal places.*

Examples.

1. What vulgar fraction is equivalent to the decimal 0.0625?

$\frac{0625}{10000}$ or $\frac{625}{10000}$; this reduced by rule under Art. 16, gives $\frac{1}{16}$; therefore $0.0625=\frac{1}{16}$.

2. What vulgar fraction is equivalent to the decimal 0.134? Ans. $\frac{67}{500}$.

3. What vulgar fraction is equivalent to the decimal 0.00125? Ans. $\frac{1}{800}$.

4. What vulgar fraction is equivalent to the decimal 0.0256? Ans. $\frac{16}{625}$.

5. What vulgar fraction is equivalent to the decimal 0.06248?

6. What vulgar fraction is equivalent to the decimal 0.001069?

CASE II.

When the decimal is a *simple repetend.*

Since $\frac{1}{9}=0.\dot{1}$, it follows that $0.\dot{2}$ must $=\frac{2}{9}$, $0.\dot{3}=\frac{3}{9}$, $0.\dot{4}=\frac{4}{9}$, and so on; therefore a simple repetend of one term, is equivalent to the vulgar fraction whose numerator is this term, and whose denominator is 9.

Again, $\frac{1}{99}=0.\dot{0}\dot{1}$, consequently $0.\dot{0}\dot{7}=\frac{7}{99}$, $0.\dot{4}\dot{5}=\frac{45}{99}$, and so on for other simple repetends of two places of figures.

In a similar manner we infer that $0.\dot{4}3\dot{2}=\frac{432}{999}$. Therefore, we have the following

RULE.

Make the repetend the numerator, and for the denominator write as many nines as there are places of decimals.

Examples.

1. What vulgar fraction is equivalent to $0.\dot{7}\dot{2}$?

$\frac{72}{99}$; this, reduced by rule under Art. 17, becomes $\frac{8}{11}$.

2. What vulgar fraction is equivalent to $0.\dot{1}2\dot{3}$?

Ans. $\frac{41}{333}$.

3. What vulgar fraction is equivalent to the repetend $0.\dot{0}2\dot{7}$? Ans. $\frac{1}{37}$.

4. What vulgar fraction is equivalent to the repetend $0.\dot{1}4285\dot{7}$? Ans. $\frac{1}{7}$.

5. What vulgar fraction is equivalent to $0.\dot{0}1234567\dot{9}$?

6. What vulgar fraction is equivalent to $0.\dot{0}12345678\dot{9}$?

CASE III.

When the decimal is a *compound repetend.*

In this case we obviously have the following

RULE.

First, find the vulgar fraction, which is equivalent to the decimal figures which precede those that circulate, by rule under Case I, *of this Article.*

Secondly, find the vulgar fraction, which is equivalent to the circulating part of the decimal, by rule under Case II, *of this Article; to the denominator of this fraction annex as many ciphers as there are decimals which precede the circulating part of the repetend; then add these two fractions together.*

Examples.

1. What vulgar fraction is equivalent to the compound repetend $0.34\dot{3}$? Ans. $\frac{34}{100}+\frac{3}{900}=\frac{309}{900}=\frac{103}{300}$.

2. What vulgar fraction is equivalent to the compound repetend $0.087\dot{8}3\dot{7}$? Ans. $\frac{13}{148}$.

3. What vulgar fraction is equivalent to $0.08\dot{3}$? Ans. $\frac{1}{12}$.

4. What vulgar fraction is equivalent to the compound repetend $0.0357\dot{1}4285\dot{7}$? Ans. $\frac{1}{28}$.

5. What vulgar fraction is equivalent to the compound repetend $0.0\dot{7}1428\dot{5}$?

6. What vulgar fraction is equivalent to $0.123\dot{4}5\dot{6}$?

CHAPTER IV.

CONTINUED FRACTIONS.

55. If we divide both numerator and denominator of the fraction $\frac{351}{965}$ by the numerator, we obtain,

$$\text{I. } \frac{351}{965}=\cfrac{1}{2+\cfrac{263}{351}}.$$

Again, performing the same operation upon the fraction $\frac{263}{351}$, we find $\frac{263}{351}=\cfrac{1}{1+\cfrac{88}{263}}$; this value of $\frac{263}{351}$ substituted in I. we get,

$$\text{II. } \frac{351}{965}=\cfrac{1}{2+\cfrac{1}{1+\cfrac{88}{263}}}.$$

Again, we find $\frac{88}{263}=\cfrac{1}{2+\cfrac{87}{88}}$; which substituted in II. gives,

$$\text{III. } \frac{351}{965}=\cfrac{1}{2+\cfrac{1}{1+\cfrac{1}{2+\cfrac{87}{88}}}}.$$

Again, $\frac{87}{88}=\cfrac{1}{1+\cfrac{1}{87}}$; this value substituted in III. we finally obtain

IV. $$\frac{351}{965}=\cfrac{1}{2+\cfrac{1}{1+\cfrac{1}{2+\cfrac{1}{1+\cfrac{1}{87}}}}}.$$

By a similar process we find that $$\frac{157}{972}=\cfrac{1}{6+\cfrac{1}{5+\cfrac{1}{4+\cfrac{1}{3+\cfrac{1}{2}}}}}.$$

Such fractions as the above are called *continued fractions.*

In the last example, the parts $\frac{1}{6}$, $\frac{1}{5}$, $\frac{1}{4}$, &c. are called the *first, second, third, &c. partial fractions.*

If we seek for the greatest common measure of the numerator and denominator of the first fraction $\frac{351}{965}$, by the rule under Art. 9, we shall obtain,

OPERATION.

```
351)965(2
    702
    ---
    263)351(1
        263
        ---
         88)263(2
            176
            ---
             87)88(1
                87
                --
                 1)87(87
                   87
                   --
                    0
```

Here we discover that the successive quotients are the same as the successive denominators of the partial fractions,

which compose the continued fraction already drawn from $\frac{251}{764}$.

Hence, to convert a vulgar fraction into a continued fraction, we have this

RULE.

Seek, by rule under Art. 9, the greatest common measure of the numerator and denominator of the given fraction; the reciprocals of the successive quotients will form the partial fractions which constitute the continued fraction required.

Examples.

1. Convert $\frac{251}{764}$ into a continued fraction.

OPERATION.

```
251)764(3
    753
    ---
     11)251(22
         22
         --
         31
         22
         --
          9)11(1
             9
             -
             2)9(4
               8
               -
               1)2(2
                 2
                 -
                 0
```

The partial fractions are $\frac{1}{3}$, $\frac{1}{22}$, $\frac{1}{1}$, $\frac{1}{4}$, $\frac{1}{2}$, therefore we shall have

$$\frac{251}{764}=\cfrac{1}{3+\cfrac{1}{22+\cfrac{1}{1+\cfrac{1}{4+\cfrac{1}{2}}}}}$$

2. What continued fraction is equivalent to $\frac{1769}{5537}$?

$$\text{Ans. } \cfrac{1}{3+\cfrac{1}{7+\cfrac{1}{1+\cfrac{1}{2+\cfrac{1}{4+\cfrac{1}{5+\cfrac{1}{1+\cfrac{1}{2.}}}}}}}}$$

3. What continued fraction is equivalent to $\frac{113}{194}$?

$$\text{Ans. } \cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{2+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{7+\cfrac{1}{2.}}}}}}}$$

4. What continued fraction is equivalent to $\frac{68}{251}$?

$$\text{Ans. } \cfrac{1}{3+\cfrac{1}{1+\cfrac{1}{2+\cfrac{1}{4+\cfrac{1}{5.}}}}}$$

---

56. Let us now endeavor to reverse the foregoing process, that is, let us seek the vulgar fraction which is equivalent to a continued fraction.

If we take the continued fraction $\cfrac{1}{2+\cfrac{1}{3+\cfrac{1}{4+\cfrac{1}{5,}}}}$

and omit all but the first partial fraction, its value will become $\frac{1}{2}=\frac{1}{2}$.

Again, omitting all but the first and second partial fractions, we find $\cfrac{1}{2+\cfrac{1}{3}}=\frac{3}{7}$.

Again, including one more partial fraction, we obtain $\cfrac{1}{2+\cfrac{1}{3+\cfrac{1}{4,}}}=\frac{13}{30}$.

When we include the whole, we find $\cfrac{1}{2+\cfrac{1}{3+\cfrac{1}{4+\cfrac{1}{5,}}}}=\frac{68}{157}$.

Our successive values obtained in this way, are $\frac{1}{2}$, $\frac{3}{7}$, $\frac{13}{30}$, and $\frac{68}{157}$.

These values may be derived in the following manner: Take the first partial fraction for the first value, multiply both numerator and denominator by the denominator of the next partial fraction, and we get $\frac{3}{6}$; if we increase this denominator by 1, it will give the second value $\frac{3}{7}$. Again, multiplying numerator and denominator by the denominator of the next partial fraction, we get $\frac{12}{28}$; if we increase this numerator by the numerator of the last value, also increase the denominator

by the denominator of the last value, we get $\frac{13}{30}$, which is the third value. Again, multiplying both numerator and denominator of this value, by the denominator of the next partial fraction, and to the respective products add the numerator and denominator of the preceding value, we obtain the last value, $\frac{68}{157}$.

This last value is the true value of the continued fraction, whilst the other values are successive approximations.

From what has been said we derive the following rule, for finding the vulgar fraction equivalent to a continued fraction.

RULE.

Consider the symbol $\frac{0}{1}$ as a fraction; then write this symbol and the first partial fraction for the two first terms of the approximate values. Multiply the numerator and denominator of the second approximate value by the denominator of the next partial fraction, and to the respective products add the numerator and denominator of the next preceding approximate value, and the result will be the succeeding approximate value. Thus continue to multiply the last approximative value, by the denominator of the succeeding partial fraction, and to the products add the numerator and denominator of the preceding approximate value, the result will be the succeeding approximate value.

Examples.

1. What vulgar fraction is equivalent to the continued fraction $\cfrac{1}{3+\cfrac{1}{2+\cfrac{1}{5+\cfrac{1}{4+\cfrac{1}{6}}}}}$?

In this example we find the successive approximate values to be $\frac{0}{1}$, $\frac{1}{3}$, $\frac{2}{7}$, $\frac{11}{38}$, $\frac{46}{159}$, and $\frac{287}{992}$.

2. What are the approximative values of the continued fraction $\cfrac{1}{3+\cfrac{1}{1+\cfrac{1}{2+\cfrac{1}{4+\cfrac{1}{5}}}}}$?

Ans. $\frac{0}{1}$, $\frac{1}{3}$, $\frac{1}{4}$, $\frac{3}{11}$, $\frac{13}{48}$, and $\frac{68}{251}$.

3. What are the approximative values of the continued fraction $\cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{2}}}}$?

Ans. $\frac{0}{1}$, $\frac{1}{2}$, $\frac{2}{5}$, $\frac{5}{12}$, $\frac{12}{29}$.

4. What are the approximative values of the continued fraction $\cfrac{1}{1+\cfrac{1}{2+\cfrac{1}{3+\cfrac{1}{4+\cfrac{1}{5+\cfrac{1}{6+\cfrac{1}{7+\cfrac{1}{8+\cfrac{1}{9}}}}}}}}}$?

Ans. $\frac{0}{1}$, $\frac{1}{1}$, $\frac{2}{3}$, $\frac{7}{10}$, $\frac{30}{43}$, $\frac{157}{225}$, $\frac{972}{1393}$, $\frac{6961}{9976}$, $\frac{56660}{81201}$, $\frac{516901}{740785}$.

5. What are the approximative values of the continued fraction $\cfrac{1}{9+\cfrac{1}{8+\cfrac{1}{7+\cfrac{1}{6+\cfrac{1}{5+\cfrac{1}{4+\cfrac{1}{3+\cfrac{1}{2}}}}}}}}$?

Ans. $\frac{0}{1}$, $\frac{1}{9}$, $\frac{8}{73}$, $\frac{57}{520}$, $\frac{350}{3193}$, $\frac{1807}{16485}$, $\frac{7578}{69133}$, $\frac{24541}{223884}$, $\frac{56660}{516901}$.

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57. We will now show the application of the foregoing principles of continued fractions, by the solution of several practical questions.

1. Express approximately the fractional part of 24 hours, by which the solar year, of 365 days 5 hours 48 minutes and 48 seconds, exceeds 365 days.

5 hours 48 minutes 48 seconds$=$20928 seconds.

24 hours$=$86400 seconds. Therefore the true value of the fraction required is $\frac{20928}{86400}=\frac{109}{450}$.

Now, converting $\frac{109}{450}$ into a continued fraction, by rule under Art. 55, we get $\frac{109}{450}=\cfrac{1}{4+\cfrac{1}{7+\cfrac{1}{1+\cfrac{1}{3+\cfrac{1}{1+\cfrac{1}{2}}}}}}$; and this re-

converted into its approximative values (by rule under Art. 56) gives $\frac{0}{1}$, $\frac{1}{4}$, $\frac{7}{29}$, $\frac{8}{33}$, $\frac{31}{128}$, $\frac{39}{161}$, $\frac{109}{450}$.

The fraction $\frac{1}{4}$, agrees with the correction introduced into the Calendar by Julius Cæsar, by means of *bissextile* or *leap-year*.

The fraction $\frac{8}{33}$ is the correction used by the Persian Astronomers, who add 8 days in every 33 years, by having seven regular leap-years, and then deferring the eighth until five years.

2. The French *metre* is 39.371 inches. Required the approximative ratio of the English foot to the *metre*.

In this example, the true ratio is $\frac{12000}{39371}$. Operating with this fraction as in the last example, we find some of the first approximative values to be $\frac{1}{3}$, $\frac{3}{10}$, $\frac{4}{13}$, $\frac{7}{23}$, $\frac{25}{82}$, $\frac{32}{105}$.

Hence, the foot is to the *metre*, as 3 to 10, nearly; a more correct ratio is as 32 to 105.

3. What are some of the approximative values of the ratio of the diameter of a circle to its circumference?

If we take the value of the circumference of the circle whose diameter is 1, to ten decimals, we have the vulgar fraction $\frac{10000000000}{31415926535}$, given to find its approximative values.

Proceeding with this, as in the former examples, we find some of the first approximative values to be $\frac{1}{3}$, $\frac{7}{22}$, $\frac{113}{355}$, &c.

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58. Continued fractions have been the means of obtaining elegant approximations to the roots of surds.

The method of converting a surd into a continued fraction is too complicated for us to explain in this place.

The square root of 2, or, what is the same thing, the ratio of the diagonal of a square to its side, is found to equal the fol-

lowing infinite continued fraction $\cfrac{1}{1+\cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{2+\&c.}}}}}$

Some of the first approximative values of this fraction are $\frac{0}{1}$, $\frac{1}{1}$, $\frac{2}{3}$, $\frac{5}{7}$, $\frac{12}{17}$, $\frac{29}{41}$, $\frac{70}{99}$, $\frac{169}{239}$, &c.

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59. We will conclude this subject by pointing out some of the many remarkable properties which the approximative values of continued fractions possess. We will refer to the values just obtained, for the ratio of the side of a square to its diagonal.

I. These values are alternately too small and too large. Thus $\frac{0}{1}$, $\frac{2}{3}$, $\frac{12}{17}$, *and* $\frac{70}{99}$, *are too small, whilst* $\frac{1}{1}$, $\frac{5}{7}$, $\frac{29}{41}$, *and* $\frac{169}{239}$, *are too large.*

II. Any of these values differ from the true value, by a quantity which is less than the reciprocal of the square of its denominator. Thus $\frac{12}{17}$, *which is the ratio much used by carpenters in cutting braces, differs from the true ratio by a quantity less than* $(\frac{1}{17})^2 = \frac{1}{289}$.

III. Any two consecutive terms of these approximative values, when reduced to a common denominator, will differ by a unit in their numerators. Thus $\frac{5}{7}$, *and* $\frac{12}{17}$, *when reduced to a common denominator, become* $\frac{85}{119}$, *and* $\frac{84}{119}$.

IV. The numerator and denominator of all approximative values of continued fractions are prime to each other; that is, they have no common measure.

CHAPTER V.

RULE OF THREE.

60. The quotient arising from dividing one quantity by another of the same kind, is called a *ratio.*

Thus, the ratio of 12 to 3, is $12 \div 3 = \frac{12}{3} = 4$.

The ratio of 15 yards to 5 yards, is $\frac{15}{5} = 3$.

From which we learn that the ratio of one number to another, is nothing more than the value of a vulgar fraction, whose numerator is the first term and denominator the last term.

The ratio of 7 days to 5 days, is $\frac{7}{5}$.

The ratio of $3\frac{1}{4}$ hours to $7\frac{1}{2}$ hours, is $3\frac{1}{4} \div 7\frac{1}{2} = \frac{13}{30}$.

When there are four quantities, of which the ratio of the first to the second is the same as that of the third to the fourth, these four quantities are said to be in *proportion.*

Thus 4, 6, 8, and 12, are in proportion, since the ratio of 4 to 6 is the same as 8 to 12. That is, $\frac{4}{6} = \frac{8}{12}$.

Hence, a proportion is nothing more than an equality of ratios.

The usual method of denoting that four terms are in proportion is by means of points or dots.

Thus, $4 : 6 :: 8 : 12$; where two points are placed between the first and second terms, and also between the third and fourth, and four points are placed between the second and third: which is read, 4 is to 6 as 8 is to 12.

The first and fourth terms of a proportion are called the *extremes.* The second and third terms are called the *means.*

The first and second constitute the *first couplet.*

The third and fourth constitute the *second couplet.*

The two terms of a couplet must be of the same name or kind; since two quantities of different kinds can not have a ratio. There can be no ratio between yards and dollars; but the numbers which represent the number of yards and dollars may have a ratio.

Since the quotient of the first divided by the second, is equal to the quotient of the third divided by the fourth, it follows that the product of the extremes is equal to the product of the means.

Hence, if we divide the product of the means by the first term, we shall obtain the fourth term.

This process of finding the fourth term by means of the other three terms, is called the *Rule of Three*, which may be thus given:

RULE.

Of the three terms which are given, one will always be of the same kind as the answer sought; this will be the third term. Then, if by the nature of the question, the answer is required to be greater than the third term, divide the greater of the two remaining terms by the less, for a ratio; but if the answer is required to be less than the third term, then divide the less of the two remaining terms by the greater, for a ratio. Having obtained the ratio, multiply it by the third term, and it will give the answer in the same denomination as was the third term.

NOTE.—Before obtaining the ratio, by means of the first two terms, we must reduce them to like denominations.

Examples.

1. If in 7 weeks there are 49 days, how many days are there in 21 weeks?

In this example the answer is required to be in days; therefore we must take 49 days for our third term. And since

in 21 weeks there must be more days than in 7 weeks, we get our ratio by dividing 21 by 7, which gives $\frac{21}{7}$; this multiplied by the third term, 49 days, gives $\frac{21\times 49}{7}=147$ days, for our answer.

2. If a person perform a journey in 20 days, by traveling 10 hours each day, how long would it take him to perform the same, if he travel 8 hours each day?

In this example our answer is required to be in days; therefore we must take 20 days for our third term. And since it will evidently take more days when he travels 8 hours each day, than it did when he traveled 10 hours each day, we must divide 10 hours by 8 hours, for our ratio, which becomes $\frac{10}{8}$; this multiplied by the third term, 20 days, gives $\frac{10\times 20}{8}$, which by canceling becomes 25 days, for the answer.

3. If $\frac{11}{13}$ of a pound of sugar cost $\frac{23}{26}$ of a shilling, how much will $\frac{9}{23}$ of a pound cost?

In this example, our third term is $\frac{23}{26}$ of a shilling. And since $\frac{9}{23}$ of a pound is less than $\frac{11}{13}$, we must obtain our ratio by dividing $\frac{9}{23}$ by $\frac{11}{13}$, which gives $\frac{9\times 13}{23\times 11}$; this multiplied by the third term, $\frac{23}{26}$ of a shilling, will give $\frac{9\times 13\times 23}{23\times 11\times 26}$. To reduce this with the least labor we must resort to the method of canceling. Thus canceling the 23, which occurs in both numerator and denominator, also 13 of the numerator against a part of the 26 of the denominator, our expression will by this means become $\frac{9}{11\times 2}=\frac{9}{22}$ of a shilling.

NOTE.—This method of canceling should be used when the nature of the question will admit, since it will always simplify very much the operation.

4. If a tree 38 feet 9 inches in height give a shadow of 49 feet 2 inches, how high is that tree which at the same time casts a shadow 71 feet 7 inches?

In this example our third term is the height of the first tree, which is 38 feet 9 inches$=38\frac{3}{4}$ feet$=\frac{155}{4}$ feet: our ratio will be obtained by dividing 71 feet 7 inches$=71\frac{7}{12}$ feet$=\frac{859}{12}$ feet, by 49 feet 2 inches$=49\frac{1}{6}$ feet$=\frac{295}{6}$ feet; which becomes $\frac{859}{12}\times\frac{6}{295}$; this multiplied by the third term $\frac{155}{4}$ gives $\frac{859\times 6\times 155}{12\times 295\times 4}$. Canceling 6 of the numerator against a part of the 12 of the denominator, also canceling 5, a factor of 155 of the numerator, against 5, a factor of 295 of the denominator, we get $\frac{859\times 31}{2\times 59\times 4}=\frac{26629}{472}=56\frac{197}{472}$ feet, for the answer.

5. If $3\frac{1}{2}$ pounds of coffee cost $2\frac{1}{3}$ shillings, how much will $10\frac{1}{6}$ pounds cost?

In this example $2\frac{1}{3}=\frac{7}{3}$ shillings must be our third term; and since $10\frac{1}{6}=\frac{61}{6}$ pounds must cost more than $3\frac{1}{2}=\frac{7}{2}$ pounds; we must divide $\frac{61}{6}$ by $\frac{7}{2}$ for the ratio; making it $\frac{61\times 2}{6\times 7}$, this multiplied by the third term, $\frac{7}{3}$ shillings, will give $\frac{61\times 2\times 7}{6\times 7\times 3}$; this becomes, after canceling, $\frac{61}{3\times 3}=6\frac{7}{9}$ shillings.

6. If $16\frac{1}{7}$ yards of calico cost $21\frac{1}{3}$ shillings, how much can be bought for $32\frac{2}{7}$ shillings? Ans. $24\frac{675}{1568}$ yards.

7. Sold a tankard for £5 6*s.* at the rate of 5*s.* 4*d.* per ounce. What was its weight? Ans. 1lb. 7oz. 17pwt. 12gr.

8. If 300 men consume 70 barrels of provisions in ten months, how much will 240 men consume in the same time? Ans. 56 barrels.

9. Gave 72 dollars for 5 barrels of fish. How much will twenty barrels cost at the same rate? Ans. 288 dollars.

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10. If it take 30 yards of carpeting, which is $\frac{3}{4}$ of a yard wide to cover a floor, how many yards, which is $1\frac{1}{4}$ yards wide, will it take to cover the same floor? Ans. 18 yards.

11. If an individual receive a salary of \$700, how much will that be for each day, counting 365 days for a year?

Ans. $1\frac{67}{73}$ dollars.

12. If the circumference of a wheel is $17\frac{1}{2}$ feet, what distance will it pass over in revolving $13\frac{1}{3}$ times?

Ans. $233\frac{1}{3}$ feet.

13. If it take $2\frac{1}{2}$ yards of cloth for 2 pair of pantaloons, and $12\frac{1}{3}$ yards for 5 coats, how many yards will it require for 7 coats and 8 pair of pantaloons? Ans. $27\frac{4}{15}$ yards.

14. If $19\frac{1}{2}$ yards of cloth cost $13\frac{1}{7}$ dollars, how many dollars will $14\frac{1}{3}$ yards cost? Ans. $9\frac{541}{819}$ dollars.

15. If 3 bushels of apples cost 13 shillings, how much will $17\frac{1}{2}$ bushels cost? Ans. £3. 15*s*. 10*d*.

16. How much must be paid for $3\frac{3}{8}$ cords of wood, at $2\frac{1}{2}$ dollars per cord? Ans. $\$8\frac{7}{16}=\$8.4375$.

17. If a board 16 feet long and 9 inches wide contain 12 square feet, how long must another board be, which is 8 inches wide, to contain the same number of square feet?

Ans. 18 feet.

18. If the freight of 7 hogsheads of melasses for 18 miles is 9 dollars, what must be paid for the freight of 21 hogsheads the same distance? Ans. 27 dollars.

19. If $\frac{3}{8}$ of a vessel is worth \$1729, how much is the whole vessel worth? Ans. $\$4610\frac{2}{3}$.

20. Lent my friend 300 dollars for 6 months; afterwards he lent me 400 dollars. How long may I keep it to balance the favor? Ans. $4\frac{1}{2}$ months.

21. If a piece of land 20 rods long and 8 rods wide make one acre, how wide must it be when it is 50 rods long to contain the same? Ans. $3\frac{1}{5}$ rods.

22. If a ship sail $90\frac{3}{8}$ miles in $7\frac{1}{2}$ hours, how many hours will it require to sail 60 miles? Ans. $4\frac{708}{723}$ hours.

23. If $\frac{3}{4}$ of an acre of land is worth 54 dollars, what is $\frac{1}{10}$ of an acre worth? Ans. \$7.20.

24. If I pay $\frac{3}{4}$ of a dollar for sawing one cord of wood, how much must I pay for sawing $5\frac{3}{7}$ cords? Ans. $\$4\frac{1}{14}$.

25. If $16\frac{1}{2}$ yards of cloth are worth $\frac{2}{3}$ of $\frac{7}{8}$ of $20\frac{1}{2}$ dollars; what is that per yard? Ans. $\$\frac{287}{396}$.

26. A man is 1700 dollars in debt, and his estate amounts to but 870 dollars. How much can he pay on the dollar?
Ans. $\$0.51\frac{3}{17}$.

27. How many yards of paper, 3 quarters wide, will paper 375 square yards? Ans. 500.

28. If a staff 5 feet long cast a shade $6\frac{1}{2}$ feet, what is the height of that steeple whose shade, at the same time, measures 150 feet? Ans. $115\frac{5}{13}$ feet.

29. If 3 paces, or common steps, of a person be equal to 2 yards, how many yards are there in 170 paces? Ans. $113\frac{1}{3}$.

30. What cost 3 cwt. of coffee at 15*d.* per pound?
Ans. £21.

31. A garrison of 540 men have provisions for 365 days. How long will those provisions last if the garrison be increased to 1127 men? Ans. $174\frac{1002}{1127}$ days.

32. What will be the tax upon £763 13*s.* at the rate of 3*s.* 4*d.* on one pound sterling?

33. What is the value of a year's rent of 547 acres of land, at the rate of 15*s.* 6*d.* per acre?

34. Allowing the French *metre* to be $3\frac{7}{25}$ feet in length, how many feet are there in 46 *metres?*

35. Suppose a certain quantity of hay will feed 70 sheep 31 days, how long would it keep 131 sheep?

36. If 9 yards of silk, which is $\frac{3}{4}$ of a yard wide, line a cloak, how many yards, that is 5 quarters wide, will it take to line the same?

37. If a barrel of beer last 10 men 16 days, how long will it last 27 men?

38. If $9\frac{3}{7}$ barrels of flour be consumed by a company in 18 days, how long will $35\frac{3}{4}$ barrels last?

39. If a mill grind $19\frac{1}{4}$ bushels of corn in 1 hour 7 minutes, in what time will it grind 100 bushels?

40. If a barrel of flour will support 13 men for 27 days, how long would it support 9 men?

41. If $\frac{5}{8}$ of an acre of land cost $13, how much can be bought for $39?

42. If $\frac{3}{16}$ of a dollar will pay for $\frac{3}{4}$ of a bushel of apples, how many bushels can be bought for $7\frac{7}{8}$ dollars?

43. If 750 barrels of cider cost $2250, how much will 419 barrels cost?

44. If $\frac{3}{15}$ of a firkin of butter is worth 85 cents, what is $\frac{3}{7}$ of a firkin worth?

45. If a staff 3 feet in length give a shadow 7 feet long, how high is that tree, which at the same time casts a shadow of 90 feet?

46. A regiment of soldiers consisting of 976 men is to be clothed; each coat to contain $2\frac{1}{2}$ yards of cloth $1\frac{5}{8}$ yards wide, and to be lined with shalloon $\frac{7}{8}$ yard wide. How many yards of shalloon will be required?

47. A person owning $\frac{3}{7}$ of a coal mine, sells $\frac{3}{4}$ of his share for \$400. What is the whole mine worth at the same ratio?

48. A and B hire a pasture for \$50, in which A pastures 13 cows, and B 12. What must each pay?

49. Suppose sound to move 1106 feet in a second; how many miles distant is a cloud, in which lightning is observed $47\frac{1}{2}$ seconds before the thunder is heard, no allowance being made for the progressive motion of light?

50. If A can mow an acre of grass in $5\frac{3}{8}$ hours, and B can mow $1\frac{1}{3}$ acres in $8\frac{1}{2}$ hours, in what time can they jointly mow $8\frac{3}{4}$ acres?

COMPOUND PROPORTION.

61. When the quantity required depends upon more than three terms, the operation of finding it is called the *rule of compound proportion*, which may be thus given.

RULE.

Among the terms given there will be but one like the answer, which we will call the odd term. The other terms will appear in pairs or couplets. Form ratios out of each couplet in the same manner as in the rule of three; then multiply all the ratios and the odd term together, and it will give the answer in the same name and denomination as the odd term.

NOTE.—Before forming ratios from the couplets they must be reduced to the same denominate value.

Examples.

1. If a person travel 300 miles in 17 days, traveling only 6 hours each day, how many miles could he have gone in 15 days, by traveling ten hours each day?

In this example the answer is required in miles, therefore our odd term is 300 miles.

The first couplet consists of days; and since in 15 days, other things being the same, he could not travel as far as in 17 days, we must divide 15 by 17, which gives $\frac{15}{17}$ for the first ratio.

The second couplet consists of hours; and since in 10 hours he could travel farther than in 6 hours, we must divide 10 by 6, which gives $\frac{10}{6}$ for the second ratio.

Multiplying these two ratios and the odd term together, we get $\frac{15\times 10\times 300}{17\times 6}$. Canceling the 6 of the denominator against a part of 300 of the numerator, it becomes $\frac{15\times 10\times 50}{17}$ $=441\frac{3}{17}$ miles, for the answer.

2. If a marble slab 10 feet long, 3 feet wide, and 3 inches thick, weigh 400 pounds, what will be the weight of another slab, of the same marble, whose length is 8 feet, width 4 feet, and thickness 5 inches?

In this example the answer is required to be in pounds; therefore 400 pounds is the odd term. The first couplet consists of the lengths, and since 8 feet in length will give less weight than 10 feet, we must divide 8 by 10, which gives $\frac{8}{10}$ for the first ratio.

The second couplet consists of the widths; and since 4 feet wide will give more weight than 3 feet, we must divide 4 by 3, which gives $\frac{4}{3}$ for the second ratio.

The third couplet consists of thicknesses; and since 5 inches thick will give more weight than 3 inches, we must divide 5 by 3, which gives $\frac{5}{3}$ for the third ratio.

Multiplying the odd term and these ratios together, we get

$\frac{8\times4\times5\times400}{10\times3\times3}$. Canceling the 10 of the denominator against a part of the 400 of the numerator, we get $\frac{8\times4\times5\times40}{3\times3}=\frac{6400}{9}=711\frac{1}{9}$ pounds, for the answer.

3. If a pile of wood 8 feet long, 4 feet wide, and 4 feet high, contain one cord of wood, how many cords are there in a pile 26 feet long, 8 feet wide, and 12 feet high?

In this example one cord is the odd term. The first couplet consists of lengths; and since 26 feet long will give more wood than 8 feet, we shall have $\frac{26}{8}$ for the first ratio.

The second couplet consists of widths; and since 8 feet wide will give more than 4 feet, we get $\frac{8}{4}$ for the second ratio.

The third couplet consists of heights; and since 12 feet high will give more wood than 4 feet, we get $\frac{12}{4}$ for the third ratio.

Multiplying these ratios and the odd term, 1 cord, together, we get $\frac{26\times8\times12}{8\times4\times4}$. Canceling the 8 of the numerator against the 8 of the denominator; also one of the 4's of the denominator against a part of the 12 of the numerator, and the factor 2, of the remaining 4 of the denominator, against the factor 2 of 26, in the numerator; our expression by this means becomes $\frac{13\times3}{2}=19\frac{1}{2}$ cords, for the answer.

4. If a man perform a journey of 1250 miles in 17 days, by traveling 13 hours a day, how many days will it take him to perform a journey of 1007 miles, by traveling 10 hours each day? Ans. $17\frac{10047}{12500}$ days.

5. If 10 cows eat 8 tons of hay in 6 weeks, how many cows will eat 56 tons in 21 weeks? Ans. 20 cows.

6. If 8 men will mow 36 acres of grass in 9 days, by working 9 hours each day, how many men will be required to mow 48 acres in 12 days, by working 12 hours each day?

Ans. 6 men.

7. If 11 men can cut 49 cords of wood in 7 days, when they work 14 hours per day, how many men will it take to cut 140 cords in 28 days, by working 10 hours each day?

Ans. 11 men.

8. If 12 ounces of wool make $2\frac{1}{2}$ yards of cloth, that is 6 quarters wide, how many pounds of wool will it take to 150 yards of cloth, 4 quarters wide?

Ans. 30 pounds.

9. If 12 men can build a wall 26 feet long, 7 feet high, and 5 feet thick, in 20 days, in how many days will 28 men build a wall 156 feet long, 10 feet high, and 3 feet thick?

Ans. $44\frac{52}{637}$ days.

10. If the wages of 6 men for 14 days be 84 dollars, what will be the wages of 9 men for 16 days?

Ans. 144 dollars.

11. If 100 men in 40 days of 10 hours each, build a wall 30 feet long, 8 feet high, and 2 feet thick, how many men must be employed to build a wall 40 feet in length, 6 feet high, and 4 feet thick, in 20 days, by working 8 hours each day?

Ans. 500 men.

12. If a pile of wood $30\frac{1}{2}$ feet long, 4 feet wide, and 6 feet high, is worth 25 dollars, how much is a pile 60 feet in length, 3 feet wide, and 4 feet high, worth?

Ans. $\$24\frac{36}{61}$.

13. If 176 bushels of corn, when corn is worth 60 cents a bushel, be given for the carriage of 120 barrels of flour 60 miles, how many bushels of corn, when corn is worth 70 cents a bushel, must be given for the carriage of 80 barrels of flour 230 miles?

14. A wall was to be built 700 yards long in 29 days; after 12 men had been employed on it for 11 days, it was found they had built only 220 yards. How many more men must be put on to finish it in the given time?

15. In how many days, working 9 hours a day, will 24 men dig a trench 420 yards long, 5 yards wide, and 3 yards deep, if 248 men, working 11 hours a day, in 5 days dig a trench 230 yards long, 3 yards wide, and 2 yards deep?

16. If 25 pears can be bought for 10 lemons, and 28 lemons for 18 pomegranates, and 1 pomegranate for 48 almonds, and 50 almonds for 70 chestnuts, and 108 chestnuts for 2½ cents, how many pears can I buy for $1.35?

17. Suppose that 50 men, by working 5 hours each day, can dig, in 55 days, 24 cellars, which are each 36 feet long, 21 feet wide, and 10 feet deep, how many men would be required to dig, in 27 days, 18 cellars, which are each 48 feet long, 28 feet wide, and 8 feet deep, provided they work only 3 hours each day?

18. If 15 men eat 13 shillings' worth of bread in 6 days, when wheat is sold at 12 shillings per bushel, in how many days will 30 men eat 43½ shillings' worth, when wheat is 10 shillings per bushel?

CHAPTER VI.

ARITHMETICAL PROGRESSION.

62. A series of numbers, which succeed each other regularly, by a common difference, are said to be in *arithmetical progression.*

When the terms are constantly increasing, the series is an *arithmetical progression ascending.*

When the terms are constantly decreasing, the series is an *arithmetical progression descending.*

Thus, 1, 3, 5, 7, 9, &c., is an ascending arithmetical progression; and 10, 8, 6, 4, 2, is a descending arithmetical progression.

In arithmetical progression there are five things to be considered:

1. *The first term.*
2. *The last term.*
3. *The common difference.*
4. *The number of terms.*
5. *The sum of all the terms.*

These quantities are so related to each other, that any three of them being given, the remaining two can be found.

Hence, there must be 20 distinct cases, arising from the different combinations of these five quantities.

To give a demonstration to all the rules of these 20 cases

would be a very difficult task for the ordinary rules of arithmetic: we will therefore content ourselves with demonstrating a few of the most important of them.

CASE I.

By our definition of an ascending arithmetical progression, it follows, that the second term is equal to the first, increased by the common difference; the third is equal to the first, increased by twice the common difference; the fourth is equal to the first, increased by three times the common difference; and so on for the succeeding terms.

Hence, when we have given the first term, the common difference, and the number of terms, to find the last term, we have this

RULE.

To the first term, add the product of the common difference into the number of terms, less one.

Examples.

1. What is the 100th term of an arithmetical progression, whose first term is 2, and common difference 3?

In this example the number of terms, less one, is 99, which, multiplied by the common difference, 3, gives 297, which added to the first term, 2, makes 299 for the 100th term.

2. What is the 50th term of the arithmetical progression, whose first term is 1, the common difference $\frac{1}{2}$? Ans. $25\frac{1}{2}$.

3. A man buys 10 sheep, giving 1 dollar for the first, 3 dollars for the second, 5 dollars for the third, and so increasing in arithmetical progression. What did the last sheep cost at that rate? Ans. 19 dollars.

4. The first term of an arithmetical progression is 2, the

common difference $\frac{1}{4}$, and the number of terms 26. What is the last term?

5. A person bought 100 yards of cloth; he gave 2*s*. 6*d*. for the first yard, 2*s*. 10*d*. for the second yard, 3*s*. 2*d*. for the third yard; and so continuing to give 4*d*. more for each yard than he gave for the preceding one. How much did he give for the last yard?

CASE II.

From the nature of an arithmetical progression, we see that the second term added to the next to the last term, is equal to the first added to the last; since the second term is as much greater than the first, as the next to the last is less than the last. After the same method of reasoning we infer that the sum of any two terms equidistant from the extremes is equal to the sum of the extremes.

Hence, it follows that the terms will average just half the sum of the extremes.

Therefore, when we have given the first term, the last term, and the number of terms, to find the sum of all the terms, we have this

RULE.

Multiply half the sum of the extremes by the number of terms.

Examples.

1. The first term of an arithmetical progression is 2, the last term is 50, and the number of terms is 17. What is the sum of all the terms?

In this example, half the sum of the extremes is $\frac{2+50}{2}=26$; this multiplied by the number of terms gives $26\times 17=442$, for the sum required.

2. The first term of an arithmetical progression is 13, the last term is 1003, and the number of terms is 100. What is the sum of the progression? Ans. 50800.

3. A person travels 25 days, going 11 miles the first day, 135 the last day; the miles which he traveled in the successive days, form an arithmetical progression. How far did he go in the 25 days? Ans. 1825 miles.

4. Bought 7 books, the prices of which are in arithmetical progression. The price of the first was 8 shillings, and the price of the last was 28 shillings. What did they all come to? Ans. £6. 6*s*.

5. What is the sum of 1000 terms of an arithmetical progression, whose first term is 7 and last term 1113?

Ans. 560000.

6. The first term of an arithmetical progression is $\frac{3}{7}$, and the last term $365\frac{4}{7}$, and the number of terms 799. What is the sum of all the terms?

7. What is the sum of 37 terms of an arithmetical progression whose first term is 10, and last term 16?

CASE III.

By case I. we see that the last term is equal to the first term, increased by the product of the common difference into the number of terms, less one.

Hence, the first term must equal the last term, diminished by the product of the common difference into the number of terms, less one.

Therefore, when we have given the last term, the number of terms, and the common difference, to find the first term we have this

RULE.

From the last term subtract the product of the common difference into the number of terms, less one.

Examples.

1. The last term of an arithmetical progression is 375, the common difference is 7, and the number of terms is 54. What is the first term?

In this example the common difference multiplied by the number of terms, less one, is $7\times 53=371$, which subtracted from the last term gives $375-371=4$, for the first term.

2. The last term of an arithmetical progression is $39\frac{1}{3}$, the common difference is $\frac{2}{3}$, and the number of terms is 59. What is the first term? Ans. $\frac{2}{3}$.

3. The last term of an arithmetical progression is $49\frac{1}{2}$, the common difference is $1\frac{1}{2}$, and the number of terms 30. What is the first term? Ans. 6.

4. The last term of an arithmetical progression is 1008, the number of terms is 252, and the common difference is 4. What is the first term?

5. A note is paid in 15 annual instalments; the payments are in arithmetical progression, whose common difference is 3; the last payment was 49 dollars. What is the first payment?

CASE IV.

From case I. we see that the product of the common difference into the number of terms, less one, is equal to the last term diminished by the first. Therefore the difference of the last and first terms, divided by the common difference, is equal to the number of terms, less one.

Hence, when we have given the first term, the last term, and the common difference, to find the number of terms, we have this

RULE.

Divide the difference of the extremes by the common difference, and to the quotient add one.

Examples.

1. The first term of an arithmetical progression is 5, the last term is 176, and the common difference 3. What is the number of terms?

In this example the difference of the extremes is $176-5=171$; this divided by the common difference, gives $\frac{171}{3}=57$; this increased by 1 becomes 58 for the number of terms required.

2. The first term of an arithmetical progression is 11, the last term 88, and the common difference 7. What is the number of terms? Ans. 12.

3. A note becomes due in annual instalments, which are in arithmetical progression, whose common difference is 3; the first payment is 7 dollars, the last payment is 49 dollars. What is the number of instalments?

CASE V.

We learn from case I. that the product of the common difference into the number of terms, less one, is equal to the last term diminished by the first. Therefore the difference of the last and first terms, divided by the number of terms, less one, will give the common difference.

Hence, when we have given the first term, the last term, and the number of terms, to find the common difference, we have this

RULE.

Divide the difference of the extremes by the number of terms, less one.

Examples.

1. The first term of an arithmetical progression is 5, the last term is 176, and the number of terms 58. What is the common difference?

In this example the difference of the extremes is 171, which divided by the number of terms, less one, becomes $\frac{171}{57}=3$, for the common difference.

2. The first term of an arithmetical progression is 17, the last term 3021, the number of terms is 752. What is the common difference? Ans. 4.

3. A person performs a journey in 17 days; the distances traveled on the successive days were in arithmetical progression; the first day he went 4 miles, and the last day he went 84. How many miles more did he go on each day, than on the preceding day? Ans. 5 miles.

4. A man has seven sons, whose ages are in arithmetical progression; the age of the eldest is 41 years, the youngest is 5 years old. How many years is the common difference of their ages?

CASE VI.

By case II. we know that the sum of all the terms of an arithmetical progression is equal to half the sum of the extremes multiplied into the number of terms; therefore the number of terms is equal to the sum of all the terms divided by half the sum of the extremes.

Hence, when we have given the first term, the last term, and the sum of all the terms, to find the number of terms, we have this

RULE.

Divide the sum of all the terms by half the sum of the extremes.

Examples.

1. The first term of an arithmetical progression is 1, the last term is 1001, and the sum of all the terms is 251001. What is the number of terms?

In this example half the sum of the extremes is $\frac{1001+1}{2}=$ 501; then dividing the sum of all the terms by this, we obtain $\frac{251001}{501}=501$, for the number of terms.

2. In a triangular field of corn, the number of hills in the successive rows are in arithmetical progression: in the first row there is but one hill, in the last row there is 81 hills; and the whole number of hills in the field is 1681. How many rows are there? Ans. 41.

3. A man bought a certain number of yards of cloth for \$152.50, giving 4 cents for the first yard, and increasing regularly on each succeeding yard, up to the last yard, for which he gave \$3.01. How many yards of cloth did he purchase? Ans. 100.

4. How many terms are there in an arithmetical progression whose first term is 5, last term 75, and sum of all the terms 440?

CASE VII.

We also infer from Case II., that the sum of all the terms, divided by half the number of terms, will give the sum of the extremes. Therefore if from the quotient, of the sum of all the terms, divided by half the number of terms, we subtract the last term, we shall have left the first term.

Hence, when we have given the last term, the number of

terms, and the sum of all the terms, to find the first term, we have this

RULE.

From the quotient of the sum of all the terms, divided by half the number of terms, subtract the last term.

Examples.

1. If the last term of an arithmetical progression is 170, the number of terms 50; and the sum of all the terms 4450, what is the first term?

In this example the sum of all the terms divided by half the number of terms is $\frac{4450}{25}=178$, from which subtracting the last term, we obtain $178-170=8$, for the first term.

2. A person wishes to discharge a debt of 1125 dollars in 18 annual payments, which shall be in arithmetical progression. How much must his first payment be, so as to bring his last payment 120 dollars? Ans. 5 dollars.

3. What is the first term of an arithmetical progression whose number of terms is 27, last term 50, and sum of all the terms 729? Ans. 4.

4. The miles which a person travels in 19 successive days form an arithmetical progression, whose last term is 80, the sum of all the terms 950. How many miles did he travel the first day?

CASE VIII.

From what has been said under Case VII., we infer that the first term subtracted from the quotient, of the sum of all the terms divided by half the number of terms, will give the last term.

Hence, when we have given the sum of all the terms, the

first term, and the number of terms, to find the last term, we have this

RULE.

From the quotient of the sum of all the terms, divided by half the number of terms, subtract the first term.

Examples.

1. If the first term of an arithmetical progression is 7, the number of terms 1000, and the sum of all the terms 560000, what is the last term?

In this example the sum of all the terms, divided by half the number of terms, gives $\frac{560000}{500}=1120$, from which subtract the first term, we get $1120-7=1113$, for the last term.

2. If the first term of an arithmetical progression is 7, the number of terms 16, and the sum of all the terms 142, what is the last term? Ans. $10\frac{3}{4}$.

3. The first term of an arithmetical progression is 13, the number of terms 100, and the sum of all the terms 50300. What is the last term?

NOTE.—The remaining cases are obtained by combining the conditions of Cases I. and II.

CASE IX.

Given the first term, the common difference, and the sum of all the terms, to find the last term.

RULE.

To twice the product of the common difference into the sum of all the terms, add the square of the first term, diminished by half the common difference; extract the square root of the sum; from this root subtract half the common difference.

Examples.

1. The first term of an arithmetical progression is 4, the common difference is 7, and the sum of all the terms is 10233. What is the last term?

In this example, twice the product of the sum of all the terms into the common difference is $2\times10233\times7=143262$; the square of the first term diminished by half the common difference is $(4-\frac{7}{2})^2=(\frac{1}{2})^2=\frac{1}{4}$; this added to 143262 gives $\frac{573049}{4}$, whose square root is $\frac{757}{2}$; from this subtracting half the common difference we get $\frac{757}{2}-\frac{7}{2}=\frac{750}{2}=375$, for the last term.

2. The first term of an arithmetical progression is $\frac{2}{3}$, the common difference is $\frac{2}{3}$, and the sum of all the terms is 1180. What is the last term? Ans. $39\frac{1}{3}$.

3. A man has several sons, whose ages are in arithmetical progression; the age of the youngest is 5 years, the common difference of their ages is 6 years, and the sum of all their ages is 161. What is the age of the eldest?

Ans. 41 years.

4. The first term of an arithmetical progression is 17, the common difference is 4, and the sum of all the terms is 114228. What is the last term?

CASE X.

Given the common difference, the number of terms, and the sum of all the terms, to find the last term:

RULE.

To the quotient of the sum of the terms, divided by the number of terms, add half the product of the common difference into the number of terms, less one.

Examples.

1. The common difference of the terms of an arithmetical progression is 6, the number of terms is 7, and the sum of all the terms is 161. What is the last term?

In this example, the sum of the terms divided by the number of terms is $\frac{161}{7}=23$. Again, the common difference multiplied into the number of terms, less one, is $6 \times 6=36$, the half of which is 18, which added to 23, gives 41, for the last term.

2. The common difference of the terms of an arithmetical progression is 7, the number of terms is 54, and the sum of all the terms is 10233. What is the last term? Ans. 375.

3. The common difference of the terms of an arithmetical progression is 6, the number of terms is 14, and the sum of all the terms is 4970. What is the last term? Ans. 394.

4. The common difference of the terms of an arithmetical progression is 17, the number of terms is 48, and the sum of all the terms is 38496. What is the last term?

CASE XI.

Given the first term, the common difference, and the number of terms, to find the sum of all the terms.

RULE.

To twice the first term, add the product of the common difference into the number of terms, less one; multiply this sum by half the number of terms.

Examples.

1. The first term of an arithmetical progression is 37, the common difference is 11, and the number of terms 99. What is the sum of all the terms?

In this example, the product of the common difference into the number of terms, less one, is $11 \times 98 = 1078$; this, added to twice the first term, gives $74 + 1078 = 1152$, which, multiplied by half the number of terms, gives 57024, for the sum of all the terms.

2. The first term of an arithmetical progression is 7, the common difference is $1\frac{1}{2}$, and the number of terms 37. What is the sum of all the terms? Ans. 1258.

3. A person buys 37 sheep, paying for them in arithmetical progression; for the first he gives 3 shillings, and increases one shilling for each succeeding one. How much did they all come to? Ans. £38, 17*s*.

4. The first term of an arithmetical progression is 13, the common difference is 9, and the number of terms is 80. What is the sum of all the terms?

CASE XII.

Given the first term, the common difference, and the last term, to find the sum of all the terms.

RULE.

Divide the difference of the squares of the last and first terms, by twice the common difference, and to this quotient add half the sum of the last and first terms.

Examples.

1. The first term of an arithmetical progression is 16, the common difference is 2, and the last term 100. What is the sum of all the terms?

In this example, the difference of the squares of the last and first terms is 9744, which, divided by twice the common difference, gives 2436; this, increased by half the sum of the last and first term, becomes 2494, for the sum of all the terms.

2. The first term of an arithmetical progression is 5, the common difference is 7, and the last term is 75. What is the sum of all the terms? Ans. 440.

3. The first term of an arithmetical progression is 8, the common difference 3, and the last term 170. What is the sum of all the terms? Ans. 4895.

4. The first term of an arithmetical progression is 12, the common difference is 6, and the last term is 104. What is the sum of all the terms?

CASE XIII.

Given the common difference, the number of terms, and the last term, to find the sum of all the terms.

RULE.

From twice the last term, subtract the product of the common difference into the number of terms, less one; multiply this remainder by half the number of terms.

Examples.

1. The common difference of the terms of an arithmetical progression is 11, the number of terms is 19, and the last term is 199. What is the sum of all the terms?

In this example the product of the common difference into the number of terms, less one, is $11\times18=198$; this, subtracted from twice the last term, gives 200, which, multiplied by half the number of terms, becomes 1900, for the sum of all the terms.

2. The common difference of the terms of an arithmetical progression is 15, the number of terms is 47, and the last term is 545. What is the sum of all the terms?

Ans. 9400.

3. The common difference of the terms of an arithmetical progression is $4\frac{1}{2}$, the number of terms is 100, and the last term is 1000. What is the sum of all the terms?

CASE XIV.

Given the first term, the number of terms, and the sum of all the terms, to find the common difference.

RULE.

From twice the sum of the terms, subtract twice the product of the first term into the number of terms; divide this remainder by the product of the number of terms into the number of terms, less one.

Examples.

1. The first term of an arithmetical progression is 21, the number of terms is 50, and the sum of all the terms is 3500. What is the common difference?

In this example, twice the product of the first term into the number of terms is 2100; which, subtracted from twice the sum of the terms, gives 4900; the number of terms multiplied into the number of terms, less one, is 2450; hence, 4900, divided by 2450, gives 2, for the common difference.

2. The first term of an arithmetical progression is $\frac{5}{7}$, the number of terms is 13, and the sum of all the terms is $139\frac{2}{7}$. What is the common difference? Ans. $1\frac{2}{3}$.

3. The first term of an arithmetical progression is $\frac{3}{4}$, the number of terms is 26, and the sum of all the terms is $60\frac{1}{8}$. What is the common difference?

CASE XV.

Given the first term, the last term, and the sum of all the terms, to find the common difference.

RULE.

Divide the difference of the squares of the last and first terms by twice the sum of all the terms, diminished by the sum of the last and first terms.

Examples.

1. The first term of an arithmetical progression is $\frac{5}{7}$, the last term is $20\frac{5}{7}$, and the sum of all the terms is $139\frac{2}{7}$. What is the common difference?

In this example, the difference of the squares of the last and first terms is $(20\frac{5}{7})^2-(\frac{5}{7})^2=428\frac{4}{7}$; twice the sum of all the terms, diminished by the sum of the last and first terms, is $257\frac{1}{7}$. Dividing $428\frac{4}{7}$ by $257\frac{1}{7}$ we get $\frac{5}{3}=1\frac{2}{3}$, for the common difference.

2. The first term of an arithmetical progression is 8, the last term is 170, and the sum of all the terms is 4895. What is the common difference? Ans. 3.

3. The first term of an arithmetical progression is 12, the last term is 104, and the sum of all the terms is 954. What is the common difference?

CASE XVI.

Given the number of terms, the last term, and the sum of all the terms, to find the common difference.

RULE.

From twice the product of the number of terms into the last term, subtract twice the sum of all the terms; divide the remainder by the product of the number of terms into the number of terms, less one.

Examples.

1. The number of terms of an arithmetical progression is 17, the last term is 50, and the sum of all the terms is 442. What is the common difference?

In this example, twice the product of the number of terms into the last term is $2\times17\times50=1700$; the product of the

number of terms into the number of terms, less one, is $17 \times 16 = 272$: also, 1700, diminished by twice the sum of all the terms, becomes 816, which, divided by 272, gives 3, for the common difference.

2. The number of terms of an arithmetical progression is 14, the last term is 14, and the sum of all the terms is 105. What is the common difference? Ans. 1.

3. The number of terms of an arithmetical progression is 7, the last term is 41, and the sum of all the terms is 138. What is the common difference?

CASE XVII.

Given the first term, the common difference, and the sum of all the terms, to find the number of terms.

RULE.

Subtract the common difference from twice the first term, divide the remainder by twice the common difference; to the square of this quotient, add the quotient of twice the sum of all the terms, divided by the common difference; extract the square root of the sum: then divide twice the first term, diminished by the common difference by twice the common difference, and subtract this quotient from the root just found.

Examples.

1. The first term of an arithmetical progression is 7, the common difference is $\frac{1}{4}$, and the sum of all the terms is 142. What is the number of terms?

In this example, the common difference, subtracted from twice the first term, gives $13\frac{3}{4}$, which, divided by twice the common difference, gives $27\frac{1}{2}$, which, squared, becomes $756\frac{1}{4}$.—Twice the sum of all the terms, divided by the common difference, gives 1136, which, added to $756\frac{1}{4}$, gives $1892\frac{1}{4}$, the square

root of which is 43½; from this, subtracting 27½, we get 16, for the number of terms.

2. The first term of an arithmetical progression is 2, the common difference is 3, and the sum of all the terms is 442. What is the number of terms? Ans. 17.

3. The first term of an arithmetical progression is ¾, the common difference is ⅛, and the sum of all the terms is 60⅛. What is the number of terms?

CASE XVIII.

Given the common difference, the last term, and the sum of all the terms, to find the number of terms.

RULE.

To twice the last term add the common difference, divide the sum by twice the common difference, square the quotient, and from this square, subtract the quotient of twice the sum of the terms divided by the common difference; extract the square root of the remainder: then subtract this root from the quotient of the sum of twice the last term and common difference, divided by twice the common difference.

Examples.

1. The common difference of the terms of an arithmetical progression is ⅓, the last term is 35½, and the sum of all the terms is 1900. What is the number of terms?

In this example, twice the last term, increased by the common difference, is 71⅓, which, divided by twice the common difference, gives 107; this, squared, becomes 11449. Again, twice the sum of all the terms, divided by the common difference, gives 11400; this, subtracted from 11449, gives 49, whose square root is 7. Subtracting this root from 107, we get 100, for the number of terms.

2. The common difference of the terms of an arithmetical progression is $\frac{1}{8}$, the last term is $3\frac{7}{8}$, and the sum of all the terms is $60\frac{1}{8}$. What is the number of terms? Ans. 26.

3. The common difference of the terms of an arithmetical progression is 1, the last term is 14, and the sum of all the terms 105. What is the number of terms?

CASE XIX.

Given the common difference, the number of terms, and the sum of all the terms, to find the first term.

RULE.

Divide the sum of the terms by the number of terms; from this quotient subtract half the product of the common difference into the number of terms, less one.

Examples.

1. The common difference of the terms of an arithmetical progression is 7, the number of terms 54, and the sum of all the terms is 10233. What is the first term?

In this example, the sum of the terms, divided by the number of terms, is $189\frac{1}{2}$; half the product of the number of terms, less one, into the common difference is $185\frac{1}{2}$, which, subtracted from $189\frac{1}{2}$, leaves 4, for the first term.

2. The common difference of the terms of an arithmetical progression is $\frac{2}{3}$, the number of terms is 59, and the sum of all the terms is 1180. What is the first term? Ans. $\frac{2}{3}$.

3. The common difference of the terms of an arithmetical progression is $1\frac{1}{2}$, the number of terms is 30, and the sum of all the terms is $832\frac{1}{2}$. What is the first term?

4. A father divides \$2000 among five sons, so that each

elder should receive $40 more than his next younger brother. What is the share of the youngest?

CASE XX.

Given the common difference, the last term, and the sum of all the terms, to find the first term.

RULE.

From the square of the last term increased by half the common difference, subtract twice the product of the common difference into the sum of all the terms; extract the square root of the remainder, and to this root add half the common difference.

Examples.

1. The common difference of the terms of an arithmetical progression is 4, the last term is 1008, and the sum of all the terms is 127512. What is the first term?

In this example, the square of the last term, increased by half the common difference, is 1020100; twice the product of the common difference into the sum of all the terms is 1020096, which, subtracted from 1020100, gives 4, the square root of which is 2; this, increased by half the common difference becomes 4, for the first term.

2. The common difference of the terms of an arithmetical progression is 3, the last term is 49, and the sum of all the terms is 420. What is the first term? Ans. 7.

3. The common difference of the terms of an arithmetical progression is 10, the last term is 1003, and the sum of all the terms is 50300. What is the first term?

CHAPTER VII.

GEOMETRICAL PROGRESSION.

63. A series of numbers which succeed each other regularly, by a constant multiplier, is called a *geometrical progression.*

This constant factor, by which the successive terms are multiplied, is called the *ratio.*

When the ratio is greater than a unit, the series is called an *ascending geometrical progression.*

When the ratio is less than a unit, the series is called a *descending geometrical progression.*

Thus 1, 3, 9, 27, 81, &c., is an ascending geometrical progression, whose ratio is 3.

And 1, $\frac{1}{4}$, $\frac{1}{16}$, $\frac{1}{64}$, &c., is a descending geometrical progression, whose ratio is $\frac{1}{4}$.

In geometrical progression, as in arithmetical progression, there are five things to be considered:

1. *The first term.*
2. *The last term.*
3. *The common ratio.*
4. *The number of terms.*
5. *The sum of all the terms.*

These quantities are so related to each other, that any three being given, the remaining two can be found.

Hence, there must be 20 distinct cases arising from the different combinations of these five quantities.

The solution of some of these cases requires a knowledge of higher principles of mathematics than can be detailed by arithmetic alone.

We will give a demonstration of the rules of some of the most important cases.

CASE I.

By the definition of a geometrical progression, it follows that the second term is equal to the first term, multiplied by the ratio; the third term is equal to the first term, multiplied by the second power of the ratio; the fourth term is equal to the first term, multiplied by the third power of the ratio; and so on, for the succeeding terms.

Hence, when we have given the first term, the ratio, and the number of terms, to find the last term, we have this

RULE.

Multiply the first term by the power of the ratio, whose exponent is one less than the number of terms.

Examples.

1. The first term of a geometrical progression is 1, the ratio is 2, and the number of terms is 7. What is the last term?

In this example, the power of the ratio, whose exponent is one less than the number of terms, is $2^6=64$, which, multiplied by the first term, 1, still remains 64, for the last term.

2. The first term of a geometrical progression is 5, the ratio is 4, and the number of terms 9. What is the last term?
Ans. 327680.

3. A person traveling, goes 5 miles the first day, 10 miles the second day, 20 miles the third day, and so on, increasing in geometrical progression. If he continue to travel in this way for 7 days, how far will he go the last day?

CASE II.

If we multiply all the terms of a geometrical progression by the ratio, we shall obtain a *new* progression, whose first term equals the second term of the *old* progression; the second term of our *new* progression will equal the third term of the *old* progression, and so on, for the succeeding terms. Hence, the sum of the *old* progression, omitting the first term, equals the sum of the *new* progression, omitting its last term. The sum of the *new* progression is equal to the *old* progression, repeated as many times as there are units in the ratio. Therefore, the difference between the *new* progression and the *old* progression, is equal to the *old* progression repeated as many times as there are units in the ratio, less one. But we also know, that the difference between these progressions is equal to the last term of the *new* progression, diminished by the first term of the *old* progression; and since the *new* progression was formed by multiplying the respective terms of the *old* progression by the ratio, it follows that the last term of the *new* progression is equal to the last term of the *old* progression, repeated as many times as there are units in the ratio. Therefore, the last term of the *new* progression, diminished by the first term of the *old* progression, is equal to the last term of the *old* progression, repeated as many times as there are units in the ratio, diminished by the first term of the *old* progression. Hence, we finally obtain this condition:

That the sum of all the terms of a geometrical progression, repeated as many times as there are units in the ratio, less one,

is equal to the last term, multiplied by the ratio, and diminished by the first term.

Hence, when we have given the first term of a geometrical progression, the last term, and the ratio, to find the sum of all the terms, we have this

RULE.

Subtract the first term from the product of the last term into the ratio; divide the remainder by the ratio, less one.

Examples.

1. The first term of a geometrical progression is 4, the last term is 78732, and the ratio is 3. What is the sum of all the terms?

In this example the first term subtracted from the product of the last term, into the ratio, is 236192, which divided by the ratio, less one, gives 118096, for the sum of all the terms.

2. The first term of a geometrical progression is 5, the last term is 327680, and the ratio is 4. What is the sum of all the terms? Ans. 436905.

3. A person sowed a peck of wheat, and used the whole crop for seed the following year; the produce of this 2d year again for seed the 3d year, and so on. If, in the last year his crop is 1048576 pecks, how many pecks did he raise in all, allowing the increase to have been in a four-fold ratio.

CASE III.

Since by case I. the last term is equal to the first term, multiplied into a power of the ratio, whose exponent is equal to the number of terms, less one, it follows that the first term is equal to the last term, divided by the power of the ratio, whose exponent is one less than the number of terms.

Hence, when we have given the last term, the ratio, and the number of terms, to find the first term, we have this

RULE.

Divide the last term by a power of the ratio, whose exponent is one less than the number of terms.

Examples.

1. The last term of a geometrical progression is 1048576, the ratio is 4, and the number of terms is 11. What is the first term?

In this example the ratio 4 raised to a power, whose index is 10, one less than the number of terms, is $4^{10}=1048576$, $\therefore$ 1048576, divided by 1048576, gives 1, for the first term.

2. A man has 6 sons, among whom he divides his estate in a geometrical progression, whose ratio is 2; the last son received $4800. How much did the first son receive?

Ans. $150.

3. A person bought 10 bushels of wheat, paying for it in geometrical progression, whose ratio is 3; the last bushel cost him $196.83. What did he give for the first bushel?

CASE IV.

We also discover from case I. that the last term divided by the first term, will give the power of the ratio, whose exponent is the number of terms, less one.

Hence, when we have given the first term, the last term, and the number of terms, to find the ratio, we have this

RULE.

Divide the last term by the first term, extract that root of the quotient which is denoted by the number of terms, less one.

Examples.

1. The first term of a geometrical progression is 1, the last term is 64, and the number of terms is 7. What is the ratio?

In this example the last term divided by the first term is 64, the number of terms, less one, is 6, ∴ we must extract the 6th root of 64; we first extract the square root, which is 8, we now extract the cube root of 8, which is 2, for the ratio.

2. In a country, during peace, the population increased every year in the same ratio, and so fast, that in the space of 5 years it became from 10000 to 14641 souls. By what ratio was the increase, yearly? Ans. $\frac{11}{10}$.

3. The first term of a geometrical progression is 4, the last term is 78732, and the number of terms is 10. What is the ratio?

CASE V.

If, in case II. we write the product of the first term, into the power of the ratio, whose exponent is the number of terms, less one, instead of the last term, as drawn from case I. we shall have the sum of all the terms, repeated as many times as there are units in the number of terms, less one, equal to the power of the ratio whose exponent is equal to the number of terms, diminished by one and multiplied by the first term.

Hence, when we have given the first term, the ratio, and the number of terms, to find the sum of all the terms, we have this

RULE.

From the power of the ratio, whose exponent is the number of terms, subtract one, divide the remainder by the ratio, less one, and multiply the quotient by the first term.

Examples.

1. The first term of a geometrical progression is 3, the ratio is 4, and the number of terms 9. What is the sum of all the terms?

In this example the ratio raised to a power whose exponent is the number of terms, is $4^9=262144$: this, diminished by one, becomes 262143, which, divided by 3, gives 87381; this, multiplied by the first term, becomes $87381\times3=262143$, for the sum of all the terms.

2. A king in India, named Sheran, wished, according to the Arabic author Asephad, that Sessa, the inventor of chess, should himself choose a reward. He requested the grains of wheat, which arise, when 1 is calculated for the first square of the board, 2 for the second square, 4 for the third, and so on; reckoning for each of the 64 squares of the board twice as many grains as for the preceding. When it was calculated, to the astonishment of the king, it was found to be an enormous number. What was it?

Ans. 18446744073709551615;

a sum which, according to a moderate calculation, could not be obtained from the whole earth, in upwards of 70 years, if all the land were employed in the cultivation of wheat.

3. A gentleman married his daughter on New Year's day, and gave her husband 1 shilling towards her portion, and was to double it on the first day of every month during the year. What was her portion?

CASE VI.

We know from case V. that the sum of all the terms multiplied by the ratio, less one, is equal to one subtracted from the power of the ratio, whose exponent is the number of terms, and this remainder multiplied by the first term.

Hence, when we have given the sum of all the terms, the number of terms, and the ratio, to find the first term, we have this

RULE.

Multiply the sum of all the terms by the ratio, less one, divide the product by the power of the ratio, whose index is the number of terms, after diminishing it by one.

Examples.

1. The sum of all the terms of a geometrical progression is 262143, the number of terms is 9, and the ratio is 4. What is the first term?

In this example the sum of all the terms multiplied by the ratio, less one, is $262143 \times 3 = 786429$; the power of the ratio, whose exponent is the number of terms, is $4^9 = 262144$, this, diminished by 1, becomes 262143; $\therefore$ 786429, divided by 262143, gives 3, for the first term.

2. The sum of all the terms of a geometrical progression is $591\frac{741}{4096}$, the number of terms is 7, and the ratio is $\frac{7}{4}$. What is the first term? Ans. 9.

3. If a debt of $4095 is discharged in 12 months, by paying sums which are in geometrical progression, the ratio of which is 2, how much was the first payment?

CASE VII.

We have shown under case II., that the sum of all the terms, multiplied by the ratio, less one, is equal to the first term subtracted from the last term into the ratio; therefore the first term is equal to the product of the ratio into the last term, diminished by the product of the ratio, less one, into the sum of all the terms.

Hence, when we have given the sum of all the terms, the last term, and the ratio, to find the first term, we have this

RULE.

Multiply the last term by the ratio, and from the product subtract the product of the sum of all the terms into the ratio, less one.

Examples.

1. The sum of all the terms of a geometrical progression is 436905, the last term is 327680, and the ratio is 4. What is the first term?

In this example we find the last term, multiplied by the ratio, to be 1310720. The product of the sum of the terms into the ratio, less one, is 1310715; ∴ 1310720—1310715=5, for the first term.

2. The sum of all the terms of a geometrical progression is 6138, the last term is 3072, and the ratio is 2. What is the first term? Ans. 6.

3. The sum of all the terms of a geometrical progression is 1860040, the last term is 1240029, and the ratio is 3. What is the first term?

CASE VIII.

From the condition under case II., we see that the ratio, multiplied into the sum of all the terms diminished by the last term, is equal to the sum of all the terms, diminished by the first term.

Hence, when we have given the first term, the last term, and the sum of all the terms, to find the ratio, we have this

RULE.

Divide the sum of all the terms diminished by the first term, by the sum of all the terms diminished by the last term.

Examples.

1. The first term of a geometrical progression is 5, the last term is 327680, and the sum of all the terms is 436905. What is the ratio?

In this example the sum of all the terms, diminished by the first term, is 436900, and the sum of all the terms, diminished by the last term, is 109225; ∴ 436900, divided by 109225, gives 4, for the ratio.

2. The first term of a geometrical progression is 6, the last term is 3072, and the sum of all the terms is 6138. What is the ratio? Ans. 2.

3. The first term of a geometrical progression is 7, the last term is 1240029, and the sum of all the terms is 1860040. What is the ratio?

NOTE.—The demonstration of the rules for the following cases have not been given; they may, however, be obtained by combining the conditions of some of the foregoing cases.

CASE IX.

Given the first term, the ratio, and the sum of all the terms, to find the last term.

RULE.

To the first term, add the product of the ratio, less one, into the sum of all the terms; divide this sum by the ratio.

Examples.

1. The first term of a geometrical progression is 4, the ratio

is 3, and the sum of all the terms is 118096. What is the last term?

In this example the product of the ratio, less one, into the sum of all the terms is 236192, which, added to the first term, gives 236196, this, divided by the ratio, gives 78732, for the last term.

2. A man bought a certain number of yards of cloth, giving 3 cents for the first yard, 6 cents for the second yard, 12 cents for the third yard, and so on, for the succeeding yards. If the whole number of yards cost $122.63, what did the last cost? Ans. $62.33.

3. A person bought a certain number of pears for £4 5*s.* 3*d.* 3*qrs.*; he gave 1 farthing for the first, 2 farthings for the second, 4 for the third, and so on, doubling each time. What did he pay for the last?

CASE X.

Given the ratio, the number of terms, and the sum of all the terms, to find the last term:

RULE.

Raise the ratio to a power, whose exponent is the number of terms, less one, multiply together this power, the sum of all the terms, and the ratio, less one; then divide this product, by one less than the power of the ratio, whose exponent is the number of terms.

Examples.

1. The ratio of the terms of a geometrical progression is 3, the number of terms is 10, and the sum of all the terms is 118096. What is the last term?

In this example the ratio raised to a power, whose expo-

nent is the number of terms, less one, is $3^9=19683$, this multiplied by the sum of all the terms, and the ratio, less one, is $19683 \times 118096 \times 2 = 4648967136$; the power of the ratio, whose exponent is the number of terms, is 59049, this, diminished by one, becomes 59048; $\therefore$ 4648967136, divided by 59048, gives 78732, for the last term.

2. The ratio of the terms of a geometrical progression is 3, the number of terms is 10, and the sum of all the terms is 295240. What is the last term? Ans. 196830.

3. The ratio of the terms of a geometrical progression is 2, the number of terms is 11, and the sum of all the terms is 20470. What is the last term?

CASE XI.

Given the first term, the number of terms, and the last term, to find the sum of all the terms.

RULE.

Extract the root, denoted by the number of terms, less one, of the last and first terms; then raise these roots to a power, whose exponent is the number of terms, then divide the difference of these powers by the difference of the roots.

Examples.

1. The first term of a geometrical progression is 1, the number of terms is 10, and the last term is 19683. What is the sum of all the terms?

In this example we must extract the 9th root of the last and first terms, which give 3 and 1 for the roots; these must be raised to the 10th power, which give 59049 and 1, the difference of which is 59048, this, divided by $3-1=2$, gives 29524, for the sum of all the terms.

2. The first term of a geometrical progression is 1, the last term is 2048, and the number of terms is 12. What is the sum of all the terms? Ans. 4095.

3. The first term of a geometrical progression is 1, the last term is 19683, and the number of terms is 10. What is the sum of all the terms?

CASE XII.

Given the ratio, the number of terms, and the last term, to find the sum of all the terms.

RULE.

Raise the ratio to a power, whose exponent is the number of terms, from this power subtract one, and multiply the remainder by the last term; divide this product by the product of the ratio, less one, into the power of the ratio, whose exponent is the number of terms, less one.

Examples.

1. The ratio of the terms of a geometrical progression is 2, the number of terms is 12, and the last term is 2048. What is the sum of all the terms?

In this example the ratio, raised to a power, whose exponent is the number of terms, is $2^{12}=4096$, this, diminished by one, becomes 4095, which, multiplied by 2048, becomes 8386560; again, the power of the ratio, whose exponent is one less than the number of terms, is 2048, which, multiplied by the ratio, less one, is not changed; $\therefore$ 8386560 divided by 2048, gives 4095, for the sum of all the terms.

2. The ratio of the terms of a geometrical progression is $\frac{3}{2}$, the number of terms is 8, and the last term is $106\frac{493}{512}$. What is the sum of all the terms? Ans. $307\frac{441}{512}$.

3. The ratio of the terms of a geometrical progression is $\frac{7}{4}$, the number of terms is 7, and the last term is $258\frac{2073}{4096}$. What is the sum of all the terms?

NOTE.—The eight remaining cases in geometrical progression, can not be solved by the ordinary processes of arithmetic, but require for their solution, a knowledge of *logarithms* and *algebraic equations*, above the second degree.

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64. When the ratio of a geometrical progression is less than a unit, the first term will be the largest, and the last term the least—the progression will, in this case, be descending; but if we consider the series of terms in a reverse order, that is, calling the last term the first, and the first the last, the progression may then be considered as ascending.

If a decreasing geometrical progression be continued to an infinite number of terms, we may neglect the last term as of no appreciable value; we can find its sum by case II., when it is modified, as follows:

Given the first term of a descending geometrical progression, and the ratio, to find the sum of all the terms, when continued to infinity.

RULE.

Divide the first term by a unit diminished by the ratio.

Examples.

1. What is the sum of all the terms of the infinite series 1, $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{8}$, &c.?

In this example a unit, diminished by the ratio, is $1-\frac{1}{2}=\frac{1}{2}$, and the first term, 1, divided by $\frac{1}{2}$, gives 2, for the sum of all the terms.

2. What is the sum of the infinite series 1, $\frac{1}{3}$, $\frac{1}{9}$, $\frac{1}{27}$, &c.?
Ans. $1\frac{1}{2}$.

3. What is the sum of the infinite series $\frac{1}{10}$, $\frac{3}{100}$, $\frac{9}{1000}$, $\frac{27}{10000}$, &c.?
Ans. $\frac{1}{7}$.

4. What is the sum of the infinite series $\frac{2}{10}$, $\frac{4}{100}$, $\frac{8}{1000}$, $\frac{16}{10000}$, &c.?
Ans. $\frac{1}{4}$.

5. What is the sum of the infinite series $\frac{1}{100}$, $\frac{3}{10000}$, $\frac{9}{1000000}$, &c.?
Ans. $\frac{1}{97}$.

6. What is the sum of the infinite series $\frac{1}{10}$, $\frac{6}{100}$, $\frac{36}{1000}$, $\frac{216}{10000}$, &c.?

7. What is the sum of the infinite series $\frac{7}{100}$, $\frac{63}{10000}$, $\frac{567}{1000000}$, &c.?

From a mere inspection of the above examples, we discover that a vulgar fraction whose numerator is 1, when converted into a decimal fraction, forms a geometrical series, whose first term agrees with the first significant figure in the decimal; the ratio is the first remainder which was obtained when dividing by the denominator.

CHAPTER VIII.

SIMPLE INTEREST.

65. *Interest* is money paid by the borrower to the lender, for the use of the money borrowed.

It is estimated at a certain *per cent. per annum*, that is, a certain number of dollars for the use of $100, for one year.

Thus, when $6 is paid for the use of $100, for one year, the interest is said to be at 6 *per cent.*

In the same manner when $5 is paid for the use of $100, for one year, the interest is said to be at 5 *per cent.;* and the same for other *rates.*

The *rate per cent.* is generally fixed by law. In the New England States the legal *rate* is 6 *per cent.*, whilst in the State of New York it is 7 *per cent.*

The sum of money borrowed, or upon which the interest is computed, is called the *principal.*

The principal, with the interest added to it, is called the *amount.*

CASE I.

To find the interest on $1, for any given time, at 6 *per cent.*

The interest on $100, for one year, at 6 per cent., being $6, it follows that the interest on $1, for one year, is $0.06; and since 2 months is $\frac{2}{12}=\frac{1}{6}$ of a year, the interest on $1, for 2 months, is $0.01; again, since 6 days is $\frac{6}{60}=\frac{1}{10}$ of 2 months, when we reckon 30 days to each month, it follows that the

interest on $1, for 6 days, is $0.001. Hence, we have the following

RULE.

Call half the number of months, CENTS; *one sixth the number of days,* MILLS.

Examples.

1. What is the interest of $1, for 7 months and 10 days, at 6 per cent.?

In this example half the number of months is 3½, which being called cents, gives $0.035, for the interest of $1, for 7 months; again, one-sixth of the number of days is 1⅔, which, being called mills, gives $0.001⅔, for the interest of $1, for 10 days; therefore, the interest for $1, for 7 months and 10 days, is $0.036⅔.

2. What is the interest of $1, for 11 months and 11 days, at 6 per cent.? Ans. $0.056⅚.

3. What is the interest of $1, for 3 years 7 months, that is, for 43 months, at 6 per cent.? Ans. $0.215.

4. What is the interest of $1, for 2 years 7 months and 9 days, at 6 per cent.? Ans. $0.1565.

5. What is the interest of $1, for 1 year 7 months and 15 days, at 6 per cent.? Ans. $0.0975.

6. What is the interest of $1, for 7 years and 9 days, at 6 per cent.? Ans. $0.4215.

7. What is the interest of $1, for 3 years 5 months and 3 days, at 6 per cent.? Ans. $0.2055.

8. What is the interest of $1, for 9 years and 3 months, at 6 per cent.?

9. What is the interest of $1, for 21 years 5 months and 6 days, at 6 per cent.?

10. What is the interest of \$1, for 7 months and 11 days, at 6 per cent.?

11. What is the interest of \$1, for 3 years and 9 months, at 6 per cent.?

CASE II.

To find the interest of any given principal, for any given time, at 6 per cent., we have this

RULE.

Find the interest on \$1, for the given time, by case I., multiply the interest thus found by the given principal.

Examples.

1. What is the interest of \$49.37, for 13 months and 15 days, at 6 per cent.?

In this example we find the interest on \$1, for 13 months and 15 days, at 6 per cent., to be \$0.0675, which, multiplied by the principal, gives \$3.332475, for the interest on \$49.37, for the given time.

2. What is the interest of \$608.62, for 1 year and 9 months, at 6 per cent.? Ans. \$63.9051.

3. What is the interest of \$341.13, for 7 years and 9 days, at 6 per cent.? Ans. \$143.786295.

4. What is the interest of \$100, for 16 years and 8 months, at 6 per cent.? Ans. \$100.

5. What is the interest of \$591.03, for 4 years 3 months and 7 days, at 6 per cent.? Ans. \$151.402185.

6. What is the interest of \$0.134, for 4 months and 3 days, at 6 per cent.? Ans. 0.002747.

7. What is the interest of \$7.50, for 7 months, at 6 per cent.?

8. What is the interest of $371.01, for 4 years and 15 days, at 6 per cent.?

9. What is the interest of $57.92, for 3 years 7 months and 9 days, at 6 per cent.?

10. What is the interest of $329, for 5 years and 13 days, at 6 per cent.?

11. What is the interest of $47.39, for 1 year and 7 months, at 6 per cent.?

CASE III.

To find the interest on any given principal, for any given time, at any given rate per cent., we have this

RULE.

Find the interest on the given principal, for the given time, at 6 per cent., by case II. Then increase, or decrease, this interest by the same part of itself, as it would be necessary to increase, or decrease, 6, in order to make it agree with the given per cent.

Examples.

1. What is the interest of $19.41, for 1 year 7 months and 13 days, at 7 per cent.?

In this example, we find by case II., that the interest of $19.41, for 1 year 7 months and 13 days, at 6 per cent., is $1.886005. Since 6, increased by its sixth part, equals 7, it will be necessary to increase the interest just found, for 6 per cent., by its sixth part, which becomes 2.200339\frac{1}{6}$, for the interest at 7 per cent.

2. What is the interest of $530, for 3 years and 6 months, at 5 per cent.? Ans. $92.75.

In this example it was necessary to decrease the interest at 6 per cent., by its sixth part.

3. What is the interest of \$5.37, for 4 years and 12 days, at 8 per cent.? Ans. \$1.73272.

In this example we increased the interest at 6 per cent. by its third part.

4. What is the interest of \$4070, for 3 months, at 9 per cent.? Ans. \$91.575.

5. What is the interest of \$3671, for 6 months, at 10 per cent.? Ans. \$183.55.

6. What is the interest of \$4920.05, for 3 months, at 4 per cent.? Ans. \$49.2005.

7. What is the interest of \$40.17, for 3 months and 18 days, at 3 per cent.? Ans. \$0.36153.

8. What is the interest of \$37.13, for 5 months and 12 days, at 4½ per cent.? Ans. \$0.7518825.

9. What is the interest of \$489, for 3 years and 4 months, at 5½ per cent.?

10. What is the interest of \$700, for 1 year and 9 months, at 7 per cent.?

NOTE.—When the principal is given in English money, we must reduce the shillings, pence, and farthings, to the decimal of a pound; and then proceed as in Federal money.

11. What is the interest of £75 13*s.* 6*d.*, for 3 years and 5 months, at 6 per cent.?

In this example, 13*s.* 6*d.*, reduced to the decimal of a pound, is 0.675, so that our principal is £75.675; the interest on £1, for 3 years and 5 months, at 6 per cent., is £0.205, which, multiplied into £75.675, gives £15.513375=£15 10*s.* 3$\frac{21}{100}$*d.*, for the interest required.

12. What is the interest of £14 5*s.* 3½*d.*, for 4 years 6 months and 14 days, at 7 per cent.? Ans. £4 10*s.* $7\frac{72}{100}$*d.*

13. What is the interest of £1 7*s.* 6*d.*, for 2 years and 6 months, at 4½ per cent.? Ans. £0 3*s.* $1\frac{1}{8}$*d.*

14. What is the interest of £105 10*s.* 6*d.*, for 9½ months, at 5 per cent.?

PARTIAL PAYMENTS.

66. When notes, bonds, or obligations, receive partial payments, or endorsements, we must use the following rule, which was given by Chancellor KENT, in the New York Chancery Reports:

RULE.

"*The rule for casting interest, when partial payments have been made, is to apply the payment, in the first place, to the discharge of the interest then due. If the payment exceed the interest, the surplus goes towards discharging the principal, and the subsequent interest is to be computed on the balance of principal remaining due. If the payment be less than the interest, the surplus of interest must not be taken to augment the principal; but interest continues on the former principal until the period when the payments, taken together, exceed the interest due, and then the surplus is to be applied towards discharging the principal; and interest is to be computed on the balance, as aforesaid.*"

Examples.

UTICA, NOV. 1, 1837.

1. For value received, I promise to pay Thomas Jones, or order, the sum of six hundred and twenty dollars, on demand, with interest. CHARLES BANK.

The following endorsements were made on this note:

1838, Oct. 6, there	was	endorsed		$61.07
1839, March 4,	"	"	"	89.03
1839, Dec. 11,	"	"	"	107.77
1840, July 20,	"	"	"	200.50

What was the balance due, Oct. 15, 1840, allowing 7 per cent. interest?

The amount of note, or principal, is	$620.000
Interest on the same, up to Oct. 6, 1838, is	40.386
Amount due on note, Oct. 6, 1838, is	660.386
The first endorsement is	61.070
	599.316
Interest from Oct. 6, 1838, to March 4, 1839, is	17.247
Amount due March 4, 1839, is	616.563
The second endorsement is	89.030
	527.533
Interest from March 4, 1839, to Dec. 11, 1839, is	28.413
	555.946
The third endorsement is	107.770
	448.176
Interest from Dec. 11, 1839, to July 20, 1840, is	19.085
	467.261
The fourth endorsement is	200.500
	266.761
Interest from July 20, 1840, to Oct. 15, 1840, is	4.410
Ans.	271.171

UTICA, May 1, 1836.

2. For value received, I promise to pay Isaac Spencer Clark, or order, three hundred and forty-nine dollars ninety-nine cents and eight mills, with interest, at 6 per cent.

JAMES N. BROWN.

Endorsements were made on this note as follows:

Dec. 25, 1836, there was paid	$49.998
June 30, 1837, " " "	4.998
Aug. 22, 1838, " " "	15.000
June 4, 1839, " " "	99.999

How much was due April 5, 1840?

The amount of the note, or principal, is		$349.998
Interest up to Dec. 25, 1836, is		13.650
		363.648
The first endorsement is		49.998
		313.650
Interest up to June 4, 1839, is		45.950
		359.600
Endorsement June 30, 1837, which is less than the interest then due,	$4.998	
Endorsement Aug. 22, 1838,	15.000	
	19.998	
This sum is still less than the interest now due.		
Endorsement June 4, 1839,	99.999	
		$119.997
This sum exceeds the interest now due.		
		239.603
Interest up to April 5, 1840, is		12.020
Amount due April 5, 1840,		251.623

UTICA, Dec. 9, 1835.

3. For value received, I promise to pay Peter Smith, or order, one hundred and eight dollars and forty-three cents, on demand, with interest, at 7 per cent. JOHN SAVBALL.

Endorsements were made as follows:

March 3, 1836, there was endorsed	$50.04
Dec. 10, 1836, " " "	13.19
May 1, 1838, " " "	50.11

How much remained due Oct. 9, 1840?

Ans. $5.844.

UTICA, Aug. 1, 1837.

4. For value received, I promise to pay W. Frederick Gould, or bearer, one hundred and forty-three dollars and fifty cents, on demand, with interest. DUDLEY FARLING.

Dec. 17, 1837, there was endorsed	$37.40
July 1, 1838, " " "	7.09
Dec. 22, 1839, " " "	13.13
Sept. 9, 1840, " " "	50.50

How much remains due Dec. 28, 1840, the interest being 7 per cent.?

5. A note of $486 is dated Sept. 7, 1831, on which,

March 22, 1832, there was paid	$125
Nov. 29, 1832, " " "	150
May 13, 1833, " " "	120

What was the balance due April 19, 1834, the interest being 7 per cent.?

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67. The principal, the rate per cent., the time, and the interest, are so related to each other, that any three of them being given, the remaining one can be found.

PROBLEM I.

Given the principal, the rate per cent., and the time, to find the interest. The rule for this problem has already been given under case III. Art. 65; it is as follows:

RULE.

Multiply the interest of $1, for the given time, and given rate per cent., by the principal.

PROBLEM II.

Given the time, the rate per cent., and the interest, to find the principal. By the reverse of the last problem we obtain this

RULE.

Divide the given interest by the interest of $1, for the given time, and given rate per cent.

Examples.

1. The interest on a certain principal, for 9 months and 10 days, at 4½ per cent., is $1.01605. What was the principal?

In this example we find the interest of $1, for 9 months 10 days, at 4½ per cent., to be $0.035; ∴ $1.01605, divided by $0.035, gives $29.03, for the principal required.

2. What principal will, in 1 year 7 months and 15 days, at 6 per cent., give $9.75 interest? Ans. $100.

3. What principal will, in 7 years and 9 days, at 6 per cent., give $16.86 interest? Ans. $40.

4. What principal will, in 3 years and 6 months, at 5 per cent., give $92.75 interest?

5. What principal will, in 3 months and 9 days, at 8 per cent., give $90 interest?

PROBLEM III.

Given the principal, the time, and the interest, to find the rate per cent.

RULE.

Divide the given interest by the interest of the given principal, for the given time, at 1 per cent.

Examples.

1. The interest of \$100, for 9 months and 10 days, is \$3.50. What is the rate per cent.?

In this example we find the interest of \$100, for 9 months and 10 days, at 6 per cent., to be \$4.66$\frac{2}{3}$. The interest at 1 per cent. is \$0.77$\frac{7}{9}$; therefore, dividing \$3.50 by \$0.77$\frac{7}{9}$, we obtain 4$\frac{1}{2}$, for the rate per cent. required.

2. At what rate per cent. will \$530, in 3 years and 6 months, give \$92.75 interest? Ans. 5 per cent.

3. At what rate per cent. will \$19.41, in 1 year 7 months and 13 days, give \$2.200339$\frac{1}{6}$ interest? Ans. 7 per cent.

4. At what rate per cent. will \$5.37, in 4 years 12 days, give \$1.73272 interest?

5. At what rate per cent. will \$4070, in 3 months, give \$91.575 interest?

PROBLEM IV.

Given the principal, the rate per cent., and the interest, to find the time.

RULE.

Divide the given interest by the interest of the given principal, for 1 year, at the given rate per cent.

Examples.

1. In what time will \$37.13, at $4\frac{1}{2}$ per cent., give \$0.7518825 interest?

In this example we find the interest of \$37.13, for 1 year, at $4\frac{1}{2}$ per cent., to be \$1.67085; ∴ dividing \$0.7518825 by \$1.67085, we get 0.45 years; this, reduced to months and days, gives 5 months and 12 days.

2. In what time will \$700, at 7 per cent., give \$85.75 interest? Ans. 1 year 9 months.

3. In what time will \$100, at 6 per cent., give \$100 interest? That is, in what time will a given principal double itself at 6 per cent. interest? Ans. $16\frac{2}{3}$ years.

4. In what time will a given principal double itself at 7 per cent.? Ans. $14\frac{2}{7}$ years.

5. In what time will a given principal double itself at 8 per cent.? Ans. $12\frac{1}{2}$ years.

6. In what time will a given principal double itself at 5 per cent.?

7. In what time will a given principal double itself at $4\frac{1}{2}$ per cent.?

The following table gives the time required for a given principal to double itself at simple interest.

Per cent.	Years.	Per cent.	Years.	Per cent.	Years.
1	100	4	25	7	$14\frac{2}{7}$
$1\frac{1}{2}$	$66\frac{2}{3}$	$4\frac{1}{2}$	$22\frac{2}{9}$	$7\frac{1}{2}$	$13\frac{1}{3}$
2	50	5	20	8	$12\frac{1}{2}$
$2\frac{1}{2}$	40	$5\frac{1}{2}$	$18\frac{2}{11}$	$8\frac{1}{2}$	$11\frac{13}{17}$
3	$33\frac{1}{3}$	6	$16\frac{2}{3}$	9	$11\frac{1}{9}$
$3\frac{1}{2}$	$28\frac{4}{7}$	$6\frac{1}{2}$	$15\frac{5}{13}$	$9\frac{1}{2}$	$10\frac{10}{19}$

DISCOUNT.

68. DISCOUNT is an allowance made for the payment of money before it is due.

The *present worth* of a debt, payable at some future time, without interest, is such a sum of money as will, in the given time, amount to the debt.

When the interest is at 6 per cent., the amount of $1, for 1 year, is $1.06; therefore the present worth of $1.06, due 1 year hence, is $1. We may also infer that the present worth of any sum for 1 year, will be as many dollars as $1.06 is contained in the given sum. Hence we have the following

RULE.

Find the amount of $1, for the given time, at the given rate per cent., then divide the sum by this amount, and it will give the present worth. Subtract the present worth from the amount, and it will give the discount.

Examples.

1. What is the present worth of $622.75, due 3 years and six months, at 5 per cent.?

In this example we find the amount of $1, for 3 years and 6 months, at 5 per cent., to be $1.175; therefore dividing $622.75 by $1.175, we get $530, for the present worth. If we subtract the present worth from the sum, we get $92.75 for the discount.

2. What is the present worth of $4161.575, due 3 months hence, at 9 per cent.? Ans. $4070.

3. What is the present worth of $7.10272, due 4 years and 12 days hence, at 8 per cent.? Ans. $5.37.

4. Sold goods for \$1500, to be paid one-half in 6 months, and the other half in 9 months; what is the present worth of the goods, interest being at 7 per cent.?

Ans. \$1437.226.

5. What is the present worth of \$50, payable at the end of 3 months, at 7 per cent.? Ans. \$49.14.

6. What is the discount on \$100, due 6 months hence, at 6 per cent.? Ans. \$2.913.

7. What is the discount on \$750, due 9 months hence, at 7 per cent.? Ans. \$37.411.

8. What is the present worth of \$3471.20, due 3 years and 9 months hence, at $4\frac{1}{2}$ per cent.? Ans. \$2970.01.

9. What is the discount of \$150, due 3 months and 18 days hence, at 6 per cent.?

10. What is the discount of \$961.13, due 1 year and 5 months hence, at 7 per cent.?

11. What is the discount of \$37.40, due at the end of 7 months, at 6 per cent.?

CHAPTER IX.

COMPOUND INTEREST.

69. When at the end of each year, the interest due is added to the principal, and the amount thus obtained is considered as a new principal, upon which the interest is cast for another year, and added to it to form a new principal for the next year, and so on to the last year, the last amount thus obtained, is called the AMOUNT AT COMPOUND INTEREST. If from this amount we subtract the original principal, we obtain the COMPOUND INTEREST.

Examples.

1. What is the compound interest of $1000, for 3 years, at 7 per cent.?

Principal,	$1000
Interest on $1000 for one year,	70
First amount, or second principal,	1070
Interest on $1070 for one year,	74.90
Second amount, or third principal,	1144.90
Interest on $1144.90 for one year,	80.143
Third amount,	1225.043
Original principal,	1000
The compound interest required,	Ans. $225.043

2. What is the compound interest of $100, for 4 years, at 6 per cent.?

Principal,	$100
Interest for first year,	6
First amount, or second principal,	106
Interest for second year,	6.36
Second amount, or third principal,	112.36
Interest for third year,	6.74
Third amount, or fourth principal,	119.10
Interest for fourth year,	7.15
Fourth amount,	126.25
Original principal,	100
Compound interest required,	Ans. $26.25

3. What is the compound interest of $630, for 4 years, at 5 per cent.? Ans. $135.769.

By carefully reviewing the above manner in which compound interest is computed, we discover that the successive amounts, which are considered as new principals, form the terms of a geometrical series, whose first term is the original principal, the ratio is the amount of $1, for one year, at the given rate per cent.; the number of terms is equal to the number of years, *plus* one.

From this we learn, that finding the amount of a given principal, for a given number of years, at a given rate per cent., consists in finding the last term of a geometrical progression, when the first term, the ratio, and the number of terms are given. This question has been solved by *Case I., of geometrical progression.*

The following table gives the amount of $1, or £1, for any number of years, not exceeding 30, at 3, 4, 5, and 6 per cent., at compound interest, the interest being compounded *yearly.*

Years.	3 per cent.	4 per cent.	5 per cent.	6 per cent.
1	1.030000	1.040000	1.050000	1.060000
2	1.060900	1.081600	1.102500	1.123600
3	1.092727	1.124864	1.157625	1.191016
4	1.125509	1.169859	1.215506	1.262477
5	1.159274	1.216653	1.276282	1.338220
6	1.194052	1.265319	1.340096	1.418519
7	1.229874	1.315932	1.407100	1.503630
8	1.266770	1.368569	1.477455	1.593848
9	1.304773	1.423312	1.551328	1.689279
10	1.343916	1.480244	1.628895	1.790848
11	1.384234	1.539454	1.710339	1.898299
12	1.425761	1.601032	1.795856	2.012196
13	1.468534	1.665074	1.885649	2.132928
14	1.512590	1.731676	1.979932	2.260904
15	1.557967	1.800944	2.078928	2.396558
16	1.604706	1.872981	2.182875	2.540352
17	1.652848	1.947900	2.292018	2.692773
18	1.702433	2.025817	2.406619	2.854339
19	1.753506	2.106849	2.526950	3.021599
20	1.806111	2.191123	2.653298	3.207135
21	1.860295	2.278768	2.785963	3.399564
22	1.916103	2.369919	3.925261	3.603537
23	1.973587	2.464716	2.071524	3.819750
24	2.032794	2.563304	3.225100	4.048935
25	2.093778	2.665836	8.386855	4.291871
26	2.156591	2.772470	3.555673	4.549383
27	2.221289	2.883369	3.733456	4.822346
28	2.287928	2.998703	3.920129	5.111687
29	2.356566	2.118651	4.116136	5.418388
30	2.427262	2.243398	4.321942	5.743491

We will now solve the following questions by means of the above table.

4. What is the amount of \$790, for 13 years, at 6 per cent.?

From our table we find the amount of \$1, for 13 years, at 6 per cent. to be \$2.132928; this multiplied by the principal, \$790, gives \$1685.013, for the amount required.

5. What is the compound interest of \$49, for 20 years, at 5 per cent.?

In this example, we find from the table, that the amount of \$1, for 20 years, at 5 per cent., is \$2.653298, which, multiplied by \$49, gives \$130.012 for the amount of \$49, from which, if we subtract \$49, we get \$81.012, for the compound interest required.

6. What is the compound interest of \$100, for 17 years, at 6 per cent.? Ans. \$169.277.

7. What is the compound interest of \$375, for 20 years, at 6 per cent.? \$827.676.

8. What is the amount of \$875, for 12 years, at 6 per cent., compound interest? Ans. \$1760.672.

9. What is the amount of \$625, for 18 years, at 5 per cent., compound interest? Ans. \$1504.137.

10. What is the amount of \$379, for 30 years, at 3 per cent., compound interest?

11. What is the amount of \$4000, for 27 years, at 4 per cent., compound interest?

NOTE.—When the interest is compounded *half-yearly*, we must take the amount of \$1, for half a year, and raise it to a power denoted by the number of half-years in the whole time; this power multiplied by the principal will give the amount. We must proceed in a similar way for any other aliquot part of a year.

12. What is the amount of $100, for 3 years, at 6 per cent. per annum, when the interest is added at the end of every 6 months?

In this example, we change the 6 per cent. to 3 per cent., and the 3 years to 6 years; we then find the tabular number to be $1.194052, which, multiplied by 100, gives $119.405, for the amount required.

13. What will £600 amount to in 6 years, at 8 per cent. compound interest, supposing the interest to be receivable half-yearly? Ans. £960 12*s.* 4¼*d.*

14. What will $890 amount to in 5 years 4 months, at 9 per cent. per annum, compound interest, the interest being added at the end of every 4 months? Ans. $1428.188.

15. What will $3705 amount to in 3 years and 3 months, at 12 per cent. per annum, compound interest, the interest being added at the end of every 3 months?

16. What will $378 amount to in 7 years and 6 months, at 6 per cent. per annum, the interest being compounded half-yearly?

17. What will $1000 amount to in 15 years, at 8 per cent. per annum, the interest being compounded half-yearly?

COMPOUND DISCOUNT.

70. *Compound discount* is an allowance made for the payment of money before it is due, on the supposition that the money draws *compound interest.*

The *present worth* of a debt payable at some future period without interest, is such a sum as being put out at compound interest, will, in the given time, at the given rate, amount to the debt.

Hence, the finding the present worth, resolves itself into the following:

Given the amount at compound interest, the time, and the rate per cent., to find the principal.

Under compound interest it was shown, that the amount was equal to the principal multiplied by the amount of $1, for one year, raised to a power whose exponent is the number of years. Hence, we have the following rule to find the principal, or present worth:

RULE.

Divide the amount by the amount of $1, for 1 year, raised to a power whose exponent is equal to the number of years.

Examples.

1. What is the present worth of $1685, due 13 years hence, allowing discount according to 6 per cent. compound interest?

From the table under Art. 69, we find that the amount of $1, for 1 year, at 6 per cent., raised to the 13th power, is $2.132928; ∴ dividing $1685 by $2.132928, gives $789.994, for the present worth required.

2. How much money must be placed out at compound interest to amount to $1000 in 20 years, the interest being 5 per cent.? Ans. 376.889.

3. What is the present worth of $3525, due in 3 years, discounting at 6 per cent., compound interest?

Ans. $2959.658.

4. What is the present worth of $350, due 5 years hence, discounting at 6 per cent., compound interest?

Ans. $261.54.

The *present worth* of a given sum of money, discounting at compound interest, is easily obtained by the following table.

This table gives the PRESENT WORTH of \$1, or £1, for any number of years, from 1 to 30, at 3, 4, 5, and 6 per cent., *compound discount.*

Years.	3 per cent.	4 per cent.	5 per cent.	6 per cent.
1	0.970874	0.961538	0.952381	0.943396
2	0.942596	0.924556	0.907029	0.889996
3	0.915142	0.888996	0.863838	0.839619
4	0.888487	0.854804	0.822702	0.792094
5	0.862609	0.821927	0.783526	0.747258
6	0.837484	0.790315	0.746215	0.704961
7	0.813092	0.759918	0.710681	0.665057
8	0.789409	0.730690	0.676839	0.627412
9	0.766417	0.702587	0.644609	0.591898
10	0.744094	0.675564	0.613913	0.558395
11	0.722421	0.649581	0.584679	0.526788
12	0.701380	0.624597	0.556837	0.496969
13	0.680958	0.600574	0.530321	0.468840
14	0.661118	0.577475	0.505068	0.442301
15	0.641862	0.555264	0.481017	0.417265
16	0.623167	0.533908	0.458112	0.393646
17	0.605016	0.513373	0.436297	0.371364
18	0.587395	0.493628	0.415521	0.350344
19	0.570286	0.474642	0.395734	0.330513
20	0.553676	0.456387	0.376889	0.311805
21	0.537549	0.438834	0.358942	0.294155
22	0.521892	0.421955	0.341850	0.277505
23	0.506692	0.405726	0.325571	0.261797
24	0.491934	0.390121	0.310068	0.246979
25	0.477606	0.375117	0.295303	0.232999
26	0.463695	0.360689	0.281241	0.219810
27	0.450189	0.346817	0.267848	0.207368
28	0.437077	0.333477	0.255094	0.195630
29	0.424346	0.320651	0.242946	0.184557
30	0.411987	0.308319	0.231377	0.174110

5. What is the present worth of $1000, due 27 years hence, discounting at 3 per cent., compound interest?

From the above table we find the present worth of $1, for 27 years, at 3 per cent., to be $0.450189; this, multiplied by 1000, gives $450.189, for the present worth required.

6. What is the present worth of $375, due 17 years hence, discounting at 4 per cent., compound interest?

Ans. $192.515.

7. What is the present worth of $672, due 13 years hence, discounting at 5 per cent., compound interest?

8. What is the present worth of $400, due 19 years hence, discounting at 6 per cent., compound interest?

9. What is the present worth of $111, due 29 years hence, discounting at 3 per cent., compound interest?

ANNUITIES.

71. *An annuity* is a fixed sum of money which is paid periodically, for a certain length of time.

CASE I.

To find the amount of an annuity, which has been forborne for a given time.

It is obvious that the last year's payment will be simply the annuity without any interest; the last but one will be the amount of the annuity for one year; the last but two will be the amount of the annuity for two years, and so on; and the sum of all these partial amounts, will give the total amount due. Now we discover that these partial amounts, or payments, form a geometrical progression, whose first term is the annuity, the ratio is the amount of $1 for 1 year, and the number of terms is equal to the number of years; therefore, the amount of an annuity is found by summing the terms of a

geometrical progression, when the first term, the number of terms, and the ratio, are given; this has been done by case V., under geometrical progression. The rule may be stated as follows:

RULE.

From the amount of \$1, *for* 1 *year, raised to a power whose exponent is equal to the number of years, subtract* \$1, *divide the remainder by the interest of* \$1, *for* 1 *year, then multiply the quotient by the annuity.*

NOTE.—The different powers of the amount of \$1, for one year, may be taken from the table under Art. 69.

Examples.

1. What is the amount of an annuity of \$200, which has been forborne 14 years, at 6 per cent., compound interest?

From table under Art. 69, we find the 14th power of the amount of \$1, for one year, at 6 per cent., to be \$2.260904; subtracting \$1, and dividing the remainder by \$0.06, the interest of \$1 for one year, we get \$21.01506, which, multiplied by \$200, the annuity gives \$4203.012, for the amount required.

2. Suppose a person, who has a salary of \$700 a year, payable quarterly, to allow it to remain unpaid for 4 years. How much would be due him, allowing quarterly compound interest at 12 per cent. per annum? Ans. \$3527.454.

3. What is due on a pension of \$150 a year, payable half-yearly, but forborne 2 years, allowing half-yearly compound interest, at 6 per cent. per annum? Ans. \$313.772.

4. What is due on a pension of \$350 a year, payable quarterly, but forborne 2½ years, allowing quarterly compound interest, at 12 per cent.?

Questions under this rule may be easily wrought by the following table.

This table shows the amount of an annuity forborne for any number of years, not exceeding 30, at 3, 4, 5, and 6 per cent., compound interest.

Years.	3 per cent.	4 per cent.	5 per cent.	6 per cent.
1	1.000000	1.000000	1.000000	1.000000
2	2.030000	2.040000	2.050000	2.060000
3	3.090900	3.121600	3.152500	3.183600
4	4.183627	4.246464	4.310125	4.374616
5	5.309136	5.416323	5.525631	5.637093
6	6.468410	6.632975	6.801913	6.975319
7	7.662462	7.898294	8.142008	8.393838
8	8.892336	9.514226	9.249109	9.897468
9	10.159106	10.582795	11.026564	11.491316
10	11.463879	12.006107	12.577893	13.180795
11	12.807796	13.486351	14.206787	14.971643
12	14.192030	15.025805	15.917127	16.869941
13	15.617790	16.626838	17.712983	18.882138
14	17.086324	18.291911	19.598632	21.015066
15	18.598914	20.023588	21.578564	23.275970
16	20.156881	21.824531	23.657492	25.672528
17	21.761588	23.697512	25.840366	28.212880
18	23.414435	25.645431	28.132385	30.905653
19	25.116868	27.671229	30.539004	33.759992
20	26.870374	29.778079	33.065954	36.785591
21	28.676486	31.969202	35.719252	39.992727
22	30.536780	34.247970	38.505214	43.392290
23	32.452884	36.617889	41.430475	46.995828
24	34.426470	39.082604	44.501999	50.815577
25	36.459264	41.645908	47.727099	54.864512
26	38.553042	44.311745	51.113454	59.156383
27	40.709634	47.084214	54.669126	63.705766
28	42.930923	49.967583	58.402583	68.528112
29	45.218850	52.966286	62.322712	73.639798
30	47.575416	56.084938	66.438847	79.058186

5. What is due on a pension of $1000, which has been forborne 27 years, at 3 per cent., compound interest?

From the above table we find the amount of an annuity of $1, for 27 years, at 3 per cent., to be $40.709634, which, multiplied by $1000, gives $40709.634, for the amount due.

6. What is the amount of an annuity of $50, which has been forborne 30 years, at 6 per cent., compound interest?

7. What is the amount of a pension of $300, which has been forborne 19 years, at 5 per cent., compound interest?

8. What is the amount of a pension of $900, which has been forborne 17 years, at 4 per cent., compound interest?

9. What is the amount of an annuity of $75, which has been forborne 13 years, at 5 per cent., compound interest?

CASE II.

To find the present worth of an annuity which is to terminate in a given number of years.

The present worth of an annuity, is obviously such a sum of money as will, at compound interest, produce an amount equal to the *amount of the annuity*. Therefore, if we find the amount of the annuity by Case I., we may consider it as the amount of a certain principal, which principal is the same as the present worth. We have already been taught how to find the present worth, by rule under *compound discount*. Hence, we have this

RULE.

First, find the amount of the annuity, as if it were in arrears for the whole time, by the aid of the table under Case I., of ANNUITIES.

Then, find the present worth of this amount for the given time and rate per cent., by the use of the table under COMPOUND DISCOUNT.

Examples.

1. What is the present worth of an annuity of $500, to continue 10 years, interest being 6 per cent.?

By the table under Case I., of annuities, we find the amount of an annuity of $1, for 10 years, at 6 per cent., to be $13.180795, this, multiplied by 500, gives $6590.3975, for the amount of the annuity.

Now, by the table under compound discount, we find the present worth of $1, for 10 years, at 6 per cent., to be $0.558395, which, multiplied by 6590.3975, gives $3680.045, for the present worth required.

2. What is the present worth of an annuity of $100, to continue 20 years, at 5 per cent. interest?

Ans. $1246.222.

The work under this Rule may be very much simplified by the use of the following table.

The following table gives the present worth of an annuity of $1, or £1, for any number of years, not exceeding 30, at 3, 4, 5, and 6 per cent.

Years.	3 per cent.	4 per cent.	5 per cent.	6 per cent.
1	0.970874	0.961538	0.952381	0.943396
2	1.913470	1.086095	1.859410	1.833393
3	2.828611	2.775091	2.723248	2.673012
4	3.717098	3.629895	3.545950	3.465106
5	4.579707	4.451822	4.329477	4.212364
6	5.417191	5.242137	5.075692	4.917324
7	6.230283	6.002055	5.786373	5.582381
8	7.019692	6.732745	6.463213	6.209794
9	7.786109	7.435332	7.107822	6.801692
10	8.530203	8.110896	7.721735	7.360087
11	9.252624	8.760477	8.306414	7.886875
12	9.954004	9.385074	8.863252	8.383844
13	10.634955	9.985648	9.393573	8.852683
14	11.296070	10.563122	9.898641	9.294984
15	11.936935	11.118387	10.379658	9.712249
16	12.561102	11.652296	10.837770	10.105895
17	13.166118	12.165669	11.274066	10.477260
18	13.753513	12.659197	11.689587	10.827603
19	14.323799	13.133839	12.085321	11.158116
20	14.877475	13.590326	12.462216	11.469921
21	15.415024	14.029160	12.821153	11.764077
22	15.936917	14.451115	13.163003	12.041582
23	16.443608	14.856842	13.488574	12.303379
24	16.935542	15.240963	13.798642	12.550358
25	17.413418	15.622080	14.093945	12.783356
26	17.876842	15.982769	14.375185	13.003166
27	18.327031	16.329586	14.643034	13.210534
28	18.764108	16.663063	14.898127	13.406164
29	19.188455	16.983715	15.141074	13.590721
30	19.600441	17.292033	15.372451	13.764831

To find the present worth of an annuity by means of this table, we must take from the table the present worth of $1, for the given time and rate per cent., and multiply it by the given annuity.

3. What is the present worth of an annuity of $27, for 9 years, at 4 per cent.?

From the table, we find the present worth of $1, for 9 years, at 4 per cent., to be $7.435332; this, multiplied by $27, gives $200.754, for the present worth.

4. What is the present worth of a pension of $75, for 15 years, at 5 per cent.? Ans. $778.474.

5. A young man purchases a farm for $924, and agrees to pay for it in the course of 7 years, paying $\frac{1}{7}$ part of the price at the end of each year. Allowing interest to be 6 per cent., how much cash, in advance, will pay the debt?

Ans. $736.874.

6. Allowing interest to be 6 per cent., how much shall I gain by paying $15 a year, for 10 years, in order to cancel a debt of $160, now due? Ans. $49.599.

7. What is the present worth of an annuity of $375, for 13 years, interest being reckoned at 4 per cent.?

8. What is the present worth of an annuity of $875, for 11 years, interest being 6 per cent.?

NOTE.—When an annuity does not commence until a given time has elapsed, or some particular event has happened, it is called a REVERSION.

CASE III.

To find the present worth of an annuity in reversion.

RULE.

Find, by the use of the table under last Case, the present worth of the annuity, from the present time up to the end of its

continuance; find, also, by the same table, its value for the time before it commences; the difference of these results will be the present worth.

Or, which amounts to the same thing:

Take the difference of the tabular numbers for these two periods, and multiply by the annuity.

Examples.

1. What is the present worth of an annuity of $200, to be continued 5 years, but not to commence till 2 years hence, interest being 6 per cent.?

By our table, we find the present worth of $1, for 7 years, at 6 per cent., to be $5.582381, the same for 2 years is $1.833393, the difference is $3.748988, which, multiplied by $200, gives $749.798, for the present worth.

2. A father leaves to his son a rent of $310 per annum, for 8 years, and the reversion of the same rent to his daughter for 14 years thereafter. What is the present worth of the legacy of each, at 6 per cent.? Ans. { Son's, $1925.036. Daughter's, $1807.854.

3. What is the present worth of a reversion of $100 a year, to commence in four years, and to continue for ten years, interest being at 6 per cent.? Ans. $582.988.

4. What is the present worth of a reversion of $800 a year, to continue 7 years, but not to commence until the end of 8 years, interest being 4 per cent.? Ans. $3508.514.

When the annuity is to continue for ever, it is obvious that its present worth will be that sum whose interest for 1 year is equal to the annuity; therefore to find the present worth of an annuity to continue for ever, we must divide the annuity by the interest of $1 for one year, at the given rate per cent.

5. How much must be paid, at present, for the title to an annuity of $1000, to commence in 7 years, and to continue for ever; interest at 6 per cent.?

Dividing $1000 by $0.06, we get for the present worth, if entered upon immediately, $16666.66⅔. From table under compound discount, we find the present worth of $1, for 7 years, at 6 per cent., to be $0.665057; this, multiplied by 16666.66⅔, gives $11084.283, for the present worth of $16666.66⅔, which is evidently the same as the present worth of the annuity.

6. What is the present worth of a reversion of $100 a year, to commence in 4 years, and to continue forever; interest being 6 per cent.? Ans. $1320.157.

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72. We will add the following tables more for curiosity than for any view to their utility.

The following table gives the time required for a given principal to double itself, at compound interest, the interest being compounded YEARLY.

Per cent.	Years.	Per cent.	Years.	Per cent.	Years.
1	69.666	4	17.673	7	10.245
1½	46.556	4½	15.748	7½	9 585
2	35.004	5	14.207	8	9.006
2½	28.071	5½	12.946	8½	8.497
3	23.450	6	11.896	9	8.043
3½	20.150	6½	11.007	9½	7.638

The following table gives the time required for a given principal to double itself, at compound interest, the interest being compounded HALF-YEARLY.

Per cent.	Years.	Per cent.	Years.	Per cent.	Years.
1	69.487	4	17.502	7	10.075
1½	46.382	4½	15.576	7½	9.914
2	34.830	5	14.036	8	8.837
2½	27.899	5½	12.775	8½	8.346
3	23.278	6	11.725	9	7.874
3½	19.977	6½	10.836	9½	7.468

The following table gives the time required for a given principal to double itself, at compound interest, the interest being compounded QUARTER-YEARLY.

Per cent.	Years.	Per cent.	Years.	Per cent.	Years.
1	69.400	4	17.415	7	9.989
1½	46.298	4½	15.490	7½	9.328
2	34.743	5	13.946	8	8.751
2½	27.812	5½	12.686	8½	8.241
3	23.191	6	11.639	9	7.788
3½	19.890	6½	10.750	9½	7.383

The following table gives the time required for a given principal to double itself, at compound interest, the interest being compounded EVERY INSTANT.

Per cent.	Years.	Per cent.	Years.	Per cent.	Years.
1	69.315	4	17.329	7	9.902
1½	46.210	4½	15.403	7½	9.242
2	34.657	5	13.863	8	8.665
2½	27.726	5½	12.603	8½	8.155
3	23.105	6	11.552	9	7.702
3½	19.804	6½	10.664	9½	7.296

The following table gives the amount of $1, or £1, for any number of years, up to 30, for 5 and 6 per cent., compound interest, the interest being compounded EVERY INSTANT.

Years.	5 per cent.	6 per cent.	Years.	5 per cent.	6 per cent.
1	1.0513	1.0618	16	2.2255	2.6116
2	1.1052	1.1275	17	2.3396	2.7731
3	1.1618	1.1972	18	2.4595	2.9446
4	1.2214	1.2712	19	2.5857	3.1267
5	1.2840	1.3498	20	2.7182	3.3201
6	1.3498	1.4333	21	2.8576	3.5253
7	1.4190	1.5219	22	3.0041	3.7433
8	1.4918	1.6161	23	3.1581	3.9748
9	1.5683	1.7160	24	3.3201	4.2206
10	1.6487	1.8221	25	3.4903	4.4815
11	1.7332	1.9348	26	3.6693	4.7587
12	1.8221	2.0544	27	3.8573	5.0529
13	1.9155	2.1771	28	4.0550	5.3653
14	2.0137	2.3163	29	4.2630	5.6971
15	2.1169	2.4596	30	4.4815	0.0492

CHAPTER X.

BANKING.

73. A BANK is an incorporated institution, created for the purpose of loaning money, receiving deposites, and dealing in exchange.

The *Stock*, or amount of money in trade, is limited by law, and owned by various individuals, who are called *stockholders*.

Banks are allowed to make notes, which are denominated *bank bills*, which circulate as money, because they are obliged to redeem them with *specie*.

It is customary for banks, in most cases, when they loan money, to take the interest in advance; that is, to deduct it from the face of the note, at the time the money is lent. The note is then said to be *discounted*.

The sum to be discounted, or the face of the note, is called the *amount*.

The interest deducted is called the *discount*.

What remains is called the *present worth*, or *proceeds*.

A note to be discounted, or bankable, must be made payable at some future time, and to the order of some person who endorses it.

It is usual for the banks to take interest for three days more than the time specified in the note; and the borrower is not obliged to make payment till those three days have expired, which are for this reason, called *days of grace*.

To find the banking discount on any sum of money, we have this

RULE.

Compute the interest (by Case III. Art. 65) on the given sum, for three days more than is specified.

Examples.

1. What is the banking discount on $1000, for three months, at 7 per cent.?

In this example we find the interest on $1, for 3 months and 3 days, at 6 per cent., to be $0.0155, which, multiplied by $1000, gives $15.50, for the discount at 6 per cent.; this, increased by its sixth part, becomes $18.08⅓, for the discount at 7 per cent., as required.

2. What is the banking discount of $150, for 6 months, at 6 per cent.? Ans. $4.575.

3. What is the banking discount of $375, for 3 months and 9 days, at 7 per cent.? Ans. $7.438.

4. What is the banking discount of $400, for 9 months, at 7 per cent.?

5. What is the banking discount of $29.30, for 7 months, at 5 per cent.?

6. What is the banking discount of $472, for 10 months, at 7 per cent.?

When the present worth of a bankable note, the time for which it is to be discounted, and the rate per cent. is given, to find the amount, we have this

RULE.

Compute the banking discount on $1, for the given time and ratio, subtract this discount from $1, then divide the present worth by the remainder, and the quotient will be the amount.

Examples.

1. What must be the amount of a bankable note, so that when discounted for 3, months at 6 per cent., shall give a present worth of $600 ?

In this example we find the banking discount on $1, for 3 months, to be $0.0155, which, subtracted from $1, gives $0.9845; ∴ dividing $600 by $0.9845, we obtain $609.446, for the required amount of the note.

2. What must be the face of a bankable note, so that when discounted for 2 months, at 7 per cent., the borrower shall receive $50 ? Ans. $50.62.

The following table gives the amount of a bankable note, so that when discounted at 5, 6, or 7 per cent., for any number of months, from 1 to 12, the present worth shall be just $1.

Months.	5 per cent.	6 per cent.	7 per cent.
1	1.004604	1.005530	1.006458
2	1.008827	1.010611	1.012402
3	1.013085	1.015744	1.018416
4	1.017380	1.020929	1.024503
5	1.021711	1.026167	1.030662
6	1.026079	1.031460	1.036896
7	1.030485	1.036807	1.043206
8	1.034929	1.042095	1.049593
9	1.039411	1.047669	1.056059
10	1.043932	1.053186	1.062605
11	1.048493	1.058761	1.069233
12	1.053093	1.064396	1.075944

We will now work some examples by the aid of the above table.

3. What must be the face of a bankable note, so that when discounted for 10 months, at 5 per cent., the present worth may be $1000 ?

Looking in the table directly under the 5 per cent., and adjacent to 10 months, we find \$1.043932, this, multiplied by \$1000, gives \$1043.932, for the face of the note required.

4. What must be the face of a bankable note, so that when discounted for 7 months, at 7 per cent., the present worth may be \$70.50 ? Ans. \$73.546.

5. What amount must I make my note, so that when discounted at the bank for 12 months, at 7 per cent., I may receive \$100 ? Ans. \$107.594.

6. What must be the amount of a note, so that when discounted at the bank for 6 months, at 6 per cent., the borrower may receive \$365 ?

7. What must be the amount of a note, so that when discounted at the bank for 9 months, at 7 per cent., the borrower may receive \$500 ?

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74. The banks, by their method of discounting, obtain a larger per cent. for their money then is obtained by the usual method of loaning money. To illustrate this, suppose A gets a note of \$1 discounted at the bank for 12 months, or one year, at 7 per cent., he receives \$0.93 ; the \$0.07 is retained by the bank, it being the interest of \$1 for one year: this \$0.07 may now be loaned to B, and its interest again withheld; and so on, for an indefinite period of terms; hence, at the end of the year, the bank will receive for its \$1, the number of dollars expressed by the sum of the terms of the following geometrical progression:

$1+\frac{7}{100}+\left(\frac{7}{100}\right)^2+\left(\frac{7}{100}\right)^3+$ &c., this, summed by rule under Art. 64, gives $\frac{100}{93}=1.0752688$. Therefore in this case the bank receives 7.52688 *per cent. per annum* for its money.

The longer the time for which they discount, the larger per cent. do they receive.

To make this appear obvious, suppose a person wished his note discounted at the bank for 14$\frac{2}{7}$ years, at 7 per cent.; in this case, the interest would equal the whole face of the note—so that the bank would withhold the whole amount, be that ever so large, and the borrower would not receive a single cent, but would nevertheless be obliged to pay to the bank, at the end of 14$\frac{2}{7}$ years, the face of the note. In this case the per cent. would be *infinite.*

If we go one step farther, and endeavor to discount a note at the bank for a longer period than 14$\frac{2}{7}$ years, at 7 per cent., we shall be obliged to pay to the bank money from our own pocket, before they would accept our note.

The following table shows the per cent. received by banks, when their notes are renewed at the end of any number of months, from 1 to 12, at 5, 6, and 7 per cent., *lawful interest.*

Months.	5 per cent.	6 per cent.	7 per cent.
1	5.138	6.200	7.272
2	5.149	6.216	7.295
3	5.160	6.232	7.317
4	5.172	6.248	7.339
5	5.183	6.264	7.362
6	5.194	6.281	7.385
7	5.205	6.298	7.408
8	5.217	6.315	7.432
9	5.228	6.332	7.456
10	5.239	6.349	7.480
11	5.251	6.366	7.503
12	5.263	6.383	7.527

NOTE.—Were it possible to renew their notes *every instant*, the respective rates per cent. would be 5.127, 6.182, and 7.251. This is the same as would be received if the interest were *added every instant.*

CHAPTER XI.

INVOLUTION.

75. INVOLUTION is the method of finding the powers of numbers.

We have already defined the power of a number to be the result arising from multiplying the same into itself continually, until the number has been used as a factor as many times as there are units in the exponent denoting the power. Thus, to obtain the cube, or third power of 7, we must use it as a factor three times, which will produce $7\times7\times7=343$.

Examples.

1. What is the square of 23?

Ans. 529.

2. What is the cube of 17?

Ans. 4913.

3. What is the 5th power of 47?

Ans. 229345007.

4. What is the 9th power of 9?

Ans. 387420489

5. What is the square of 22667121?

Ans. 513798374428641.

6. What is the square of 625?

Ans. 390625.

7. What is the cube of 48?

Ans. 110592.

8. What is the cube of 681472?

Ans. 316478381828866048.

9. What is the square of 0.75?

Ans. 0.5625.

10. What is the cube of 0.65?

Ans. 0.274625.

11. What is the square of $8\frac{1}{2}$?

Ans. $72\frac{1}{4}$.

EVOLUTION.

76. Evolution is the reverse of involution. It explains the method of resolving a number into equal factors, which factors are called *roots*.

When the number is resolved into two equal factors, this factor is called the *square root* of the number.

When a number is resolved into three equal factors, the factor is called the *cube root* of the number.

The operation of resolving a number into two equal factors is called the *extraction of the square root*.

EXTRACTION OF THE SQUARE ROOT.

77. If we square 48 by the usual rule, we get $48^2=2304$. But if instead of 48, we use $40+8$, we shall find, by actual multiplication,

$$\begin{array}{r} 40+8 \\ \underline{40+8} \\ 1600+320 \\ \underline{+320+64} \\ 1600+640+64 \end{array}$$

for the square of $40+8=48$. Now to reverse this operation,

that is, to extract the square root of $1600+640+64$, we proceed as follows:

We take the square root of 1600, which is 40; this is the first part of the root; its square being subtracted from $1600+640+64$, leaves the remainder $640+64$. We see that 640, divided by twice 40, or 80, gives 8 for a quotient, which is the second part of the root required.

CASE I.

From the above process, we deduce the following rule for the extraction of the square root of a whole number.

RULE.

I. Point off the given number into periods of two figures each, counting from the right towards the left. When the number of figures is odd, it is evident that the left-hand, or first period, will consist of but one figure.

II. Find the greatest square in the first period, and place its root at the right of the number, in the form of a quotient figure in division. Subtract the square of this root from the first period, and to the remainder annex the second period; the result will be the FIRST DIVIDEND.

III. Double the root already found, and place it on the left of the number, for the FIRST TRIAL DIVISOR. *See how many times this trial divisor, with a cipher annexed, is contained in the dividend, the quotient figure will be the second figure of the root; this must be placed at the right of the* TRIAL DIVISOR; *the result will be the* TRUE DIVISOR. *Multiply the true divisor by this second figure of the root, and subtract the product from the dividend, and to the remainder annex the next period, for a* SECOND DIVIDEND.

IV. To the last TRUE DIVISOR, *add the last figure of the root, for a new* TRIAL DIVISOR, *and continue to operate as before, until all the periods have been brought down.*

Examples.

1. What is the square root of 531441?

	53'14'41 (729 root
7	49
First trial divisor, 14	414 first dividend,
first true divisor, 142	284
second trial divisor, 144	13041 second dividend,
second true divisor, 1449	13041
	0

2. What is the square root of 11390625?

3	11'39'06'25(3375
63	9
667	239
6745	189
	5006
	4669
	33725
	33725

In the first example, we exhibited the trial divisors, as well as the true divisors, but in the second example we adhered more closely to our rule, and placed the succeeding figures of the root at the right of the trial divisors, without again writing them down.

3. What is the square root of 11019960576?

```
  1          1'10'19'96'05'76(104976
  204        1
  2089       -----
  20987       1019
  209946       816
               -----
               20396
               18801
               ------
                159505
                146909
                -------
                 1259676
                 1259676
                 -------
                       0
```

4. What is the square root of 16983563041?

Ans. 130321.

5. What is the square root of 79792266297612001?

Ans. 282475249.

6. What is the square root of 852891037441?

Ans. 923521.

7. What is the square root of 61917364224?

Ans. 248832.

8. What is the square root of 13422659310152401?

Ans. 115856201.

CASE II.

To extract the square root of a decimal fraction, or of a number consisting partly of a whole number, and partly of a decimal value, we have this

RULE.

I. Annex one cipher, if necessary, so that the number of decimals shall be even.

II. Then point off the decimals into periods of two figures each, counting from the unit's place towards the right. If there are whole numbers they must be pointed off as in Case I. Then extract the root, as in Case I.

NOTE.—If the given number has not an exact root, there will be a remainder after all the periods have been brought down, in which case the operation may be extended by forming new periods of ciphers.

Examples.

1. What is the square root of 3486.784401?

Ans. 59.049.

2. What is the square root of 25.62890625?

Ans. 5.0625.

3. What is the square root of 6.5536?

Ans. 2.56.

4. What is the square root of 0.00390625?

Ans. 0.0625.

5. What is the square root of 17?

Ans. 4.123, nearly.

6. What is the square root of 37.5?

Ans. 6.123, nearly.

7. What is the square root of 0.0000012321?

Ans. 0.00111.

8. What is the square root of 0.0011943936?

Ans. 0.03456.

9. What is the square root of 60.481729?

Ans. 7.777.

10. What is the square root of 0.2455?

11. What is the square root of 100.00756?

CASE III.

To extract the square root of a vulgar fraction, or mixed number, we have this

RULE.

I. Reduce the vulgar fraction, or mixed number, to its simplest fractional form.

II. Then extract the square root of the numerator and denominator separately, if they have exact roots; but when they have not, reduce the fraction to a decimal, and proceed as in Case II.

Examples.

1. What is the square root of $\frac{256}{324}$?

Ans. $\frac{8}{9}$.

2. What is the square root of $\frac{10125}{12005}$?

Ans. $\frac{45}{49}$.

3. What is the square root of $4\frac{21}{25}$?

Ans. $2\frac{1}{5}$.

4. What is the square root of $\frac{5}{9}$ of $\frac{4}{5}$ of $\frac{4}{7}$ of $\frac{7}{9}$?

Ans. $\frac{4}{9}$.

5. What is the square root of $4\frac{1}{9}$?

Ans. 2.027, nearly.

6. What is the square root of $\frac{11}{17}$?

Ans. 0.8044, nearly.

7. What is the square root of $\frac{13}{49}$?

8. What is the square root of $\frac{1}{361}$?

9. What is the square root of $\frac{1234}{4321}$?

10. What is the square root of $\frac{123444}{447789}$?

CASE IV.

When there are many figures required in the root, we may, after obtaining one more than half the number required, find the rest by dividing the remainder by the last TRUE DIVISOR, deprived of its right-hand figure. This division should be performed according to the abridged method, as explained under Art. 42, page 56.

Examples.

1. What is the square root of 11 to 16 decimals?

3	11(3.3166247903553998.
63	9
661	2
6626	189
66326	11
663322	661
6633244	439
66332487	39756
66332494|9	4144
	397956
	16444
	1326644
	317756
	26532976
	5242624
	464327409
	59934991
	5969924541
	23574559
	19899748
	3674811
	3316625
	358186
	331662
	26524
	19900
	6624
	5970
	654
	597
	57
	53
	4

In the preceding example, after obtaining 9 figures of the root, by the usual rule, we had for the remainder 23574559, the last true divisor was 66332494|9, when deprived of its right-hand figure. We then divide this remainder by this divisor, according to the method of *abridged division of decimals*, Art. 42, page 56, and obtained the remaining 8 figures of the root.

2. What is the square root of 3 to 10 decimals?

Ans. 1.7320508076.

3. What is the square root of 0.00008876684 to 10 places of decimals?

Ans. 0.0094216155.

4. What is the square root of 0.8867081118724 to 10 places of decimals?

Ans. 0.9416517994.

5. What is the square root of 3.14159265 to 8 places of decimals?

6. What is the square root of 2 to 9 places of decimals?

7. What is the square root of 100 to 15 places of decimals?

8. What is the square root of 0.365 to 7 places of decimals?

EXAMPLES INVOLVING THE PRINCIPLES OF THE SQUARE ROOT.

78. *A triangle* is a figure having three sides, and consequently three angles.

When one of the angles is right, like the corner of a square, the triangle is called a *right-angled triangle.* In this case the side opposite the right angle is called the *hypotenuse.*

It is an established proposition of geometry, that the square

of the hypotenuse is equal to the sum of the squares of the other two sides.

From the above proposition, it follows that the square of the hypotenuse, diminished by the square of one of the sides, equals the square of the other side.

By means of these properties, it follows that two sides of a right-angled triangle being given, the third side can be found.

Examples.

1. How long must a ladder be, to reach the top of a house, 40 feet high, when the foot of it is 30 feet from the house?

In this example it is obvious that the ladder forms the hypotenuse of a right-angled triangle, whose sides are 30 and 40 feet respectively. Therefore the square of the length of the ladder must equal the sum of the squares of 30 and 40.

$$30^2 = 900$$
$$40^2 = 1600$$
$$\sqrt{2500} = 50 = \text{the length of the ladder.}$$

2. Suppose a ladder, 100 feet long, to be placed 60 feet from the roots of a tree, how far up the tree will the top of the ladder reach?

Ans. 80 feet.

3. Two persons start from the same place, and go, the one due north, 50 miles, the other due west, 80 miles. How far apart are they?

Ans. 94.34 miles, nearly.

4. What is the distance through the opposite corners of a square yard?

Ans. 4.24264 feet, nearly.

5. The distance between the lower ends of two equal raf-

ters, in the different sides of a roof, is 32 feet, and the height of the ridge above the foot of the rafters is 12 feet. What is the length of a rafter?

6. What is the distance measured through the centre of a cube, from one corner to its opposite corner, the cube being 3 feet, or one yard, on a side?

We know, from the principles of geometry, that all similar surfaces, or areas, are to each other as the squares of their like dimensions.

7. Suppose we have two circular pieces of land, the one 100 feet in diameter, the other 20 feet in diameter. How much more land is there in the larger than in the smaller?

By the above principle of geometry it follows, that the quantity of land in the two circles, must be as the squares of the diameters, that is, as 100^2 to 20^2, or as 25 to 1. Hence, there is 25 times as much in the one piece, as there is in the other.

8. Two persons, the one 6 feet high, the other 5 feet: now suppose they are both well proportioned in all respects; how much more cloth will it take to make a suit of clothes for the first, than for the second?

Ans. { It will require $1\frac{11}{25}$ times as much for the first as for the second.

9. Suppose by observation, it is found that 4 gallons of water flow through a circular orifice of 1 inch in diameter in one minute. How many gallons would, under similar circumstances, be discharged through an orifice of 3 inches in diameter, in the same length of time?

Ans. 36.

10. What must be the circumference of a circular pond, which shall contain $\frac{1}{16}$ part as much surface as a pond $13\frac{1}{2}$ miles in circumference?

Ans. $3\frac{3}{8}$ miles.

11. Required the width and depth of a rectangular box, whose length is 3 feet, which shall contain 30000 solid inches; the width being to the depth as 2 to 3.

12. What length of thread is required to wind spirally around a cylinder, 2 feet in circumference and 3 feet in length, so as to go but once around?

It is evident that if the cylinder be developed, or placed upon a plane, and caused to roll once over, that the convex surface of the cylinder will give a rectangle, whose width is 2 feet, and length 3 feet; at the same time the thread will form its diagonal. Hence, the length of the thread is $\sqrt{4+9}=\sqrt{13}=3.60555$ feet.

13. Seven men purchase a grinding stone, of 60 inches in diameter. What part of the diameter must each grind off, so as to have $\frac{1}{7}$ of the whole stone?

SOLUTION.

In this question, we disregard the thickness of the stone.

After the first one has ground off his share, the remaining stone will be $\frac{6}{7}$ of the original stone. Therefore its diameter will be $60\sqrt{\frac{6}{7}}=\frac{60}{7}\sqrt{42}=55.54922$, nearly.

The diameter, after the 2d one has ground off his share, will be $60\sqrt{\frac{5}{7}}=\frac{60}{7}\sqrt{35}=50.70925$, nearly.

The diameter, after the 3d one has ground off his share, will be $60\sqrt{\frac{4}{7}}=\frac{60}{7}\sqrt{28}=45.35574$, nearly.

The diameter, after the 4th one has ground off his share, will be $60\sqrt{\frac{3}{7}}=\frac{60}{7}\sqrt{21}=39.27923$, nearly.

The diameter, after the 5th one has ground off his share, will be $60\sqrt{\frac{2}{7}}=\frac{60}{7}\sqrt{14}=32.07135$, nearly.

The diameter, after the 6th one has ground off his share, will be $60\sqrt{\frac{1}{7}}=\frac{60}{7}\sqrt{7}=22.67786$, nearly.

Hence, the parts of the diameter ground off are as follows :

				inches, nearly.
The 1st	ground	off	60.00000−55.54922=	4.45078
2d	"	"	55.54922−50.70925=	4.83997
3d	"	"	50.70925−45.35574=	5.35351
4th	"	"	45.35574−39.27923=	6.07651
5th	"	"	39.27923−32.07135=	7.20788
6th	"	"	32.07135−22.67786=	9.39349
7th	"	"	22.67786 =	22.67786
			Proof,	60.00000

EXTRACTION OF THE CUBE ROOT.

79. If we cube 45 by the usual process, we find $45^3 =$ 91125.

If, instead of 45, we take its equal, 40+5, and then cube it by actual multiplication, as explained under Art. 4, we shall have

$$
\begin{array}{rrrrr}
 & & 45= & 40+ & 5 \\
 & & & 40+ & 5 \\
\hline
 & & & 200+ & 25 \\
 & & 1600+ & 200 & \\
\hline
 & 45^2= & 1600+ & 400+ & 25 \\
 & & & 40+ & 5 \\
\hline
 & & 8000+ & 2000+ & 125 \\
 & 64000+ & 16000+ & 1000 & \\
\hline
45^3= & 64000+ & 24000+ & 3000+ & 125
\end{array}
$$

Now, to reverse this process, that is, to extract the cube root of 64000+24000+3000+125, we proceed as follows :

I. We first find the cube root of 64000 to be 40, which we place to the right of the number, in the form of a quotient in division, for the first part of the root sought.

We also place it on the left of the number, in a column headed 1st col.; we next multiply it into itself, and place the result in a column headed 2d col.; this last result, also multiplied by 40, gives 64000, which we subtract from the number, and obtain the remainder, 24000+3000+125, which we will call the FIRST DIVIDEND.

II. We obtain the second term of the 1st column by adding the first term to itself, the result being multiplied by this first term, and added to the first term of the 2d column, gives its second term. Again, adding this first term to the second term of the 1st column, we get its third term.

III. We seek how many times the second term of the 2d column, is contained in the first dividend, or simply how many times it is contained in its firstpart, 24000, which gives 5, for the second part of the root.

IV. Finally, we add this 5 to the last term of the 1st column, whose result, multiplied by 5 and added to the last term of the 2d column, gives its third term; which, multiplied by 5, gives 27125=24000+3000+125.

1st col.	2d col.	number,	root.
40	1600	64000+24000+3000+125	(40+5
80	4800	64000	
120	5425	24000+3000+125	=27125
125			27125
			0

This work can be written in a more condensed form, as follows, where the ciphers upon the right have been omitted.

1st col.	2d col.	number, root.
		91125 (45
4	16	64
8	48	27125
12	5425	27125
125		0

CASE I.

From the preceding operation we may draw the following rule for extracting the cube root of a whole number.

RULE.

I. Since the cube of any number can not have more than three times as many places of figures as the number, we must separate the number into periods of three figures each, counting from the unit's place towards the left. When the number of figures is not divisible by 3, the left-hand period will contain less than 3 figures.

II. Seek the greatest cube of the first, or left-hand period, place its root at the right of the number, after the manner of a quotient in division; also place it to the left of the number, for the first term of a column, marked 1ST COL. *Then multiply it into itself, and place the product for the first term of a column, marked* 2D COL. *Again, multiply this last result by the same figure, and subtract the product from the number, and to the remainder annex the next period, and it will give the* FIRST DIVIDEND. *This same figure must be added to the first term of the 1st column; the sum will be its second term, which must be multiplied by the same figure, and the product added to the first term of the 2d column; the sum will be its second term, which we shall name the* FIRST TRIAL DIVISOR.

The same figure of the root must be added to the second term of the 1st column, to form its third term.

III. See how many times the trial divisor, with two ciphers annexed, is contained in the dividend; the quotient figure will be the second figure of the root, which must be placed at the right of the first figure; also annex it to the third term of the 1st column, and multiply the result by this second figure, and add

the product, after advancing it two places to the right, to the last term of the 2d column. Again, multiply this last result by this second figure of the root, and subtract the product from the dividend, and to the remainder annex the next period, for a NEW DIVIDEND.

Proceed with this second figure of the root, precisely as was done with the first figure; and so continue until all the periods have been brought down.

Examples.

1. Extract the cube root of 387420489.

OPERATION.

1ST COL.	2D COL.	number, root.
		387′420′489(729
7	49	343
14	147 first trial divisor	44420 1st div.
212	15124	30248
214	15552 2d trial divisor	14172489 2d div.
2169	1574721	14172489
		0

EXPLANATION.

The greatest cube of the first period, 387, is 343, whose root is 7, which we place to the right of the number, for the first figure of the root sought. We also place it for the first term of the first column, which, multiplied into itself, gives $7\times7=49$, for the first term of the 2d column, which, in turn, multiplied by 7, gives $49\times7=343$, which, subtracted from the first period, 387, leaves the remainder 44, to which, annexing the next period, 420, we get 44420, for the *first dividend.*

Again, adding 7 to the first term, 7, of 1st column, we get $7+7=14$, for the second term of the 1st column, which, mul-

tiplied by 7, gives $14 \times 7 = 98$; this, added to the first term of the 2d column, gives 147, for the second term of the 2d column, or the *first trial divisor*.

Again, adding 7 to the second term of the 1st column, we get $14 + 7 = 21$, for the third term of the 1st column.

The *trial divisor*, with two ciphers annexed, becomes 14700, which is contained 3 times in the *first dividend*, 44420. But the *trial divisor* being less than the true divisor, it will sometimes give too large a quotient figure; such is the case in this present example, where 2 is the second figure of the root.

This second figure 2 of the root, annexed to the third term of the 1st column, gives 212, which, multiplied by 2, gives 424, which, being advanced two places to the right, must be added to 147, the last term of the 2d column. The sum 15124 will form the third term of the 2d column, which, multiplied by 2, gives $15124 \times 2 = 30248$, which, subtracted from the first dividend, leaves 14172, for the remainder, to which annexing the next period, 489, we get 14172489, for the *second dividend.*

Again, to the last term, 212, of the 1st column, adding 2, we get 214 for the next term, which, multiplied by 2, gives 428, which, added to 15124, gives 15552, for the *second trial divisor.* Again, adding 2 to 214, we get 216 for the fifth term of the 1st column.

The *second trial divisor*, with two ciphers annexed, becomes 1555200, which is contained 9 times in the *second dividend*, 14172489; therefore 9 is the third figure of the root, which, annexed to 216, gives 2169 for the last term of the 1st column, which, multiplied by 9, gives 19521, which, advanced two places to the right, and then added to 15552, gives 1574721; this multiplied by 9, gives 14172489, which, subtracted from the *second dividend*, leaves no remaider.

2. What is the cube root of 913517247483640899 ?

OPERATION.

1st col.	2d col.	number,	root.
		913'517'247'483'640'899	(970299
9	81	729	
18	243	184517	
277	26239	183673	
284	28227	844247483	
29102	282328204	564656408	
29104	282386412	279591075640	
291069	28241260821	254171347389	
291078	28243880523	25419728251899	
2910879	2824414250211	25419728251899	
		0	

3. What is the cube root of 10077696 ?

Ans. 216.

4. What is the cube root of 2357947691 ?

Ans. 1331.

5. What is the cube root of 42875 ?

Ans. 35.

6. What is the cube root of 117649 ?

Ans. 49.

7. What is the cube root of 350356403707485209 ?

Ans. 704969.

8. What is the cube root of 7508468627929687 5?

Ans. 421875.

9. What is the cube root of 7256313856 ?

Ans. 1936.

10. What is the cube root of 106868920913284608 ?

Ans. 474552.

CASE II.

To extract the cube root of a decimal fraction, or of a number consisting partly of a whole number and partly of a decimal value, we have this

RULE.

I. Annex ciphers to the decimals, if necessary, so that the whole number may be divisible by 3.

II. Separate the decimals into periods of three figures each, counting from the decimal point towards the right, and proceed as in whole numbers.

NOTE—If the given number has not an exact root, there will be a remainder after all the periods have been brought down. The process may be continued by annexing ciphers for new periods.

Examples.

1. What is the cube root of 0.469640999917?
Ans. 0.7773.
2. What is the cube root of 18.609625?
Ans. 2.65.
3. What is the cube root of 1.25992105?
Ans. 1.08005974.
4. What is the cube root of 2?
Ans. 1.25992105.
5. What is the cube root of 9?
Ans. 2.080083823.
6. What is the cube root of 3?
Ans. 1.442249.
7. What is the cube root of 1860867?
8. What is the cube root of 987654321?
9. What is the cube root of 1.23456789?

CASE III.

To extract the cube root of a vulgar fraction, or mixed number, we have this

RULE.

I. Reduce the fraction, or mixed number, to its simplest fractional form.

II. Extract the cube root of the numerator and denominator separately, if they have exact roots, but when they have not, reduce the fraction to a decimal, and then extract the root by Case II.

Examples.

1. What is the cube root of $\frac{2197}{4913}$? Ans. $\frac{13}{17}$.
2. What is the cube root of $\frac{85169}{170723}$? Ans. $\frac{23}{29}$.
3. What is the cube root of $17\frac{1}{8}$? Ans. 2.577, nearly.
4. What is the cube root of $5\frac{1}{7}$? Ans. 1.726, nearly.
5. What is the cube root of $\frac{81}{99}$? Ans. 0.9353, nearly.
6. What is the cube root of $\frac{2}{3}$? Ans. 0.8736, nearly.
7. What is the cube root of $47\frac{1}{8}$?
8. What is the cube root of $101\frac{1}{7}$?
9. What is the cube root of $9\frac{3}{8}$?

CASE IV.

When there are many decimal places required in the root, we may, after obtaining one more decimal figure than half the required number, find the rest by dividing the remainder by the last term of the SECOND COLUMN.

Before dividing, we can omit from the right of the divisor so many figures as to leave but one more than the number of additional figures required in the root; observing to omit from the right of the dividend one figure less than was omitted in the divisor. The division must then be performed according to the abridged method, as explained under Art. 42, page 56.

Examples.

1. What is the cube root of 7 carried to 9 decimal places?

1st col.	2d col.	root.
		7(1.912931182
1	1	1
2	3	6
39	651	5859
48	1083	141
571	108871	108871
572	109443	32129
5732	10955764	21911528
5734	10967232	10217472
57369	1097239521	9875155689
57378	1097755923	342316311
573873	10977|7313919	329331941757
		12984|369243
		10978
		2006
		1098
		908
		878
		30
		22
		8

In this example we proceed in the usual way, until we obtain 1.91293, the remainder was 12984369243; the last term of the second column was 109777313919; therefore we must obtain 4 more figures by dividing 12984369243 by 109777313919; but since we wish but 4 more figures, they may be obtained with equal accuracy by dividing 12984 by 10977, which gives the remaining figures 1182.

2. Extract the cube root of $\frac{1}{4}$=0.25 to 13 decimal places.

1st col.	2d col.	
		0.250(0.6299605249474
6	36	216
12	108	34
182	11164	22328
184	11532	11672
1869	1170021	10530189
1878	1186923	1141811
18879	118862211	1069759899
18888	119032203	72051101
188976	11904354156	71426124936
188982	11905488048	624976064
18898805	119054974974025	595274874870125
		29701189129875
		2381099
		589019
		476220
		112799
		107149
		5650
		4762
		888
		833
		55
		47
		8

In this example, after obtaining 7 decimal figures in the root by the usual process, the remainder was 29701189129875, and the last term in the 2*d column* was 119054974974025; and since we wish but 6 figures by division, we reject 7 figures from the right of the remainder, and 8 figures from the right of the term of the *second column;* and then divide by

the rule for abridging the work, and obtain the remaining figures of the root.

3. Extract the cube root of 9 to 9 decimals.

1st col.	2d col.	
		9 (2.080083823
2	4	8
4	12	1
608	124864	998912
616	129792	1088
624008	1297|96992064	1038375936512
		4962|4063488
		3894
		1068
		1038
		30
		26
		4
		4
		0

4. What is the cube root of $15\frac{2}{3}$ to 5 decimal places?

Ans. 2.50222.

5. What is the cube root of $\frac{1}{3.14159265}$ to 8 decimals?

Ans. 0.68278406.

6. What is the cube root of 0.0000031502374 to 13 decimals?

Ans. 0.0146593402919.

7. What is the cube root of $\frac{1}{2}$ to 21 decimals?

Ans. 0.793700525984099737376.

8. What is the cube root of $\frac{1}{7}$ to 10 decimals?

9. What is the cube root of $\frac{14}{15}$ to 7 decimals?

10. What is the cube root of $\frac{1}{365}$ to 8 decimals.

EXAMPLES INVOLVING THE PRINCIPLES OF THE CUBE ROOT.

80. It is an established theorem of geometry, that all similar solids are to each other as the cubes of their like dimensions.

1. If a cannon ball, 3 inches in diameter, weighs 8 pounds, what will a ball of the same metal weigh, whose diameter is 4 inches?

By the above theorem, we have $3^3 : 4^3 :: 8$ pounds : $18\frac{26}{27}$ pounds, for the answer.

2. What is the side of a cube, which will contain as much as a chest 8 feet 3 inches long, 3 feet wide, and 2 feet 7 inches deep?

3. Suppose the diameter of the sun is 887681 miles, the diameter of the earth 7912 miles, how many times greater in bulk is the sun than the earth?

Ans. 1412251 times, nearly.

4. Suppose the diameter of the moon to be 2160 miles, how many times greater in bulk is the sun than the moon?

5. How many cubic quarter inches can be made out of a cubic inch?

Ans. 64.

6. Required the dimensions of a rectangular box, which shall contain 20000 solid inches, the length, breadth, and depth, being to each other as 4, 3, and 2.

7. Four ladies purchased a ball of exceeding fine thread, 3 inches in diameter. What portion of the diameter must each wind off so as to share of the thread equally?

SOLUTION.

After the first one had wound off her share, the ball which

remained would contain $\frac{3}{4}$ as much thread as it did in the first place. Therefore its diameter was $3\sqrt[3]{\frac{3}{4}}=\frac{3}{2}\sqrt[3]{6}=2.72568$ inches, nearly.

The diameter, after the second one had wound off her share, was $3\sqrt[3]{\frac{1}{2}}=\frac{3}{2}\sqrt[3]{4}=2.38110$ inches, nearly.

The diameter, after the third one had wound off her share, was $3\sqrt[3]{\frac{1}{4}}=\frac{3}{2}\sqrt[3]{2}=1.88988$ inches, nearly.

Hence, the portions of the diameter which they must wind off are as follows:

		inches, nearly.
The 1st lady must wind off	3.00000 − 2.72568	= 0.27432
2d " " " "	2.72568 − 2.38110	= 0.34458
3d " " " "	2.38110 − 1.88988	= 0.49122
4th " " " "	1.88988	= 1.88988
	Proof,	3.00000

ROOTS OF ALL POWERS.

81. Whenever the index, denoting the root required, is a composite number, the root can be found by successive extractions of the roots denoted by the prime factors of the original index.

Thus, the 4th root can be found by extracting the 2d root twice in succession.

The 6th root is obtained by extracting the 3d root of the 2d root.

The 8th root is found by extracting the 2d root three times in succession.

When the index denoting the root is a prime, we must have some direct method of obtaining the root.

By a similar train of reasoning, as was used in deducing the *rule for the cube root*, we determine, in general, for any root, the following

RULE.

I. Point the number off into periods of as many figures each as there are units in the index, denoting the root.

II. Find by trial, the figure of the first period, which will be the first figure of the root; place this figure to the left, in a column called the FIRST COLUMN. *Then multiply it by itself, and place the product for the first term of the* SECOND COLUMN. *This, multiplied by the same figure, will give the first term of the* THIRD COLUMN. *Thus continue until the number of columns is one less than the units in the index, denoting the root.*

Multiply the term in the LAST COLUMN *by the same figure, and subtract the product from the first period, and to the remainder bring down the next period, and it will form the* FIRST DIVIDEND.

Again, add this same figure to the term of the FIRST COLUMN, *multiply the sum by the same figure, and add the product to the term of* SECOND COLUMN; *which, in turn, must be multiplied by the same figure, and added to the term of* THIRD COLUMN, *and so on till we reach the* LAST COLUMN; *the term of which will form the* FIRST TRIAL DIVISOR.

Again, beginning with the FIRST COLUMN, *repeat the above process until we reach the column next to the last. And so continue to do until we obtain as many terms in the* FIRST COLUMN *as there are units in the index, denoting the root; observing in each successive operation to terminate on the column of the next inferior order.*

III. Seek how many times the FIRST TRIAL DIVISOR, *when there are annexed to it as many ciphers, less one, as there are*

units in the index, is contained in the FIRST DIVIDEND, *the quotient figure will be the second figure of the root. Then proceed with this figure the same as was done with the first figure; observing to advance the terms of the different columns as many places to the right, as the number expressing the order of the column; that is, advancing the terms of the* FIRST COLUMN *one place, those of the* SECOND COLUMN *two places, and so for the succeeding columns.*

After completing the requisite number of terms in the different columns, by means of this second figure of the root, then proceed to obtain the third figure of the root, in the same way as the second figure was obtained; and in this way the operation can be continued until all the periods are brought down. If there is still a remainder, the process can be extended by forming periods of ciphers.

Examples.

1. What is the fifth root of 36936242722357?

OPERATION.

1ST COL.	2D COL.	3D COL.	4TH COL.	root.
5	25	125	625	3693'62427'22357(517
10	75	500	3125	3125
15	150	1250	32525251	56862427
20	250	1275251	33826005	32525251
251	25251	1300754	347673946051	2433717622357
252	25503	1326510		2433717622357
253	25756	1344842293		0
254	26010			
2557	2618899			

2. What is the 7th root of 2? Ans. 1.10409, nearly.

3. What is the 1th root of 11? Ans. 1.24357, nearly.

4. What is the 5th root of 5?

5. What is the 7th root of 12311715481324093447

OPERATION.

1st	2d	3d	4th	5th	6th	
COL.	COL.	COL.	COL.	COL.	COL.	12311'7154813'2409344)384
3	9	27	81	243	729	2187
6	27	108	405	1458	5103	101247154813
9	54	270	1215	5103	11568197824	92545582592
12	90	540	2835	808149728	21076554688	8701572212409344
15	135	945	37231216,	1188544608	2175393055310233 6	8701572212409344
18	189	1110152	47549360	1663938528		0
218	20644	1289768	59424240	169343966275584		
226	22452	1484360	72979760			
234	24324	1694440	737528368896			
242	26260	1920520				
250	28260	1932692224				
258	30324					
2664	3043056					

CHAPTER XII.

DUODECIMALS.

82. In decimals, we have seen that the figures decrease in in a tenfold ratio, from the left towards the right.

In DUODECIMALS, this decrement goes on in a twelve fold ratio.

The different denominations are the *foot*, ($f.$,) the *prime*, or inch, ($'$,) the *second*, ($''$,) the *third*, ($'''$,) the *fourth*, ($''''$,) the *fifth*, ($'''''$,) and so on.

Thus, $7f., 6', 3'', 4''', 5''''$, is read, 7 feet, 6 primes, 3 seconds, 4 thirds, 5 fourths.

The accents, used to distinguish the denominations below feet, are called *indices*.

Taking the foot for the unit, we have the following relations:

$1' = \frac{1}{12}$ of 1 foot,
$1'' = \frac{1}{12}$ of $\frac{1}{12}$ of 1 foot $= \frac{1}{144}$ of 1 foot,
$1''' = \frac{1}{12}$ of $\frac{1}{12}$ of $\frac{1}{12}$ of 1 foot $= \frac{1}{1728}$ of 1 foot,
$1'''' = \frac{1}{12}$ of $\frac{1}{12}$ of $\frac{1}{12}$ of $\frac{1}{12}$ of 1 foot $= \frac{1}{20736}$ of 1 foot,
&c. &c. &c. &c.

Addition and subtraction of duodecimals, are performed like addition and subtraction of other compound numbers, remembering that 12 of any denomination make one of the next greater denomination.

MULTIPLICATION OF DUODECIMALS.

83. Suppose we wish to multiply $14f.\ 7'$ by $2f.\ 3'$, we should proceed as follows:

$$\begin{array}{rrr} & 14f. & 7' \\ & 2f. & 3' \\ \hline 3f. & 7' & 9'' \\ 29f. & 2' & \\ \hline 32f. & 9' & 9'' \end{array}$$

Ans. $32f.\ 9'\ 9''=32f.+\frac{9}{12}$ of a foot $+\frac{9}{144}$ of a foot.

EXPLANATION.

We began on the right-hand, and multiplied the multiplicand through, first by the primes of the multiplier, then by the feet of the multiplier, thus: $3'\times7'=\frac{3}{12}\times\frac{7}{12}=\frac{21}{144}$ of a ft. which is $21''=1'\ 9''$; we write down the $9''$, and carry the $1'$ to the next product; again, $14f.\times3'=14\times\frac{3}{12}=\frac{42}{12}$ of a foot, which is $42'$; now adding in the $1'$, which was to carry from the last product, we have $43'=3f.\ 7'$, which we write down, thus finishing the first line of products.

Again, we have $2f.\times7'=2\times\frac{7}{12}=\frac{14}{12}$ of a foot, which is $14'=1f.\ 2'$; we write the $2'$ under the seconds of the last line, and carry $1f.$ to the next product; $2f.\times14f.=28f.$ to which, adding in the $1f.$, which was to carry from the last product, we have $29f.$, which we place underneath the feet of the last line. Taking the sum, we find $32f.\ 9'\ 9''$, for the answer.

From the above we infer, *that if we consider the index of the feet to be* 0, *then the denomination of the product will be denoted by the sum of the indices, representing the factors.*

Thus, *feet* by *feet*, produces *feet*; *feet* by *primes*, produces *primes*; *primes* by *primes*, produces *seconds*, &c., &c.

Hence, to multiply a number consisting of feet, inches, seconds, &c., by another number consisting of like qualities, we have this

RULE.

Place the several terms of the multiplier under the corresponding ones of the multiplicand. Beginning at the right-hand, multiply the several terms of the multiplicand by the several terms of the multiplier successively, placing the right-hand term of each of the partial products under its multiplier; then add the partial products together, observing to carry one for every twelve, both in multiplying and adding. The sum of the partial products will be the answer.

Examples.

1. What is the product of 3*f*. 7′ 2″ by 7*f*. 6′ 3″?

	3*f*.	7′	2″		
	7*f*.	6′	3″		
			10″	9‴	6⁗
	1*f*.	9′	7″	0‴	
	25*f*.	2′	2″		
Ans.	27*f*.	0′	7″	9‴	6⁗

2. What is the product of 7*f*. 6′ 4″ by 2*f*. 3 5″?

	7*f*.	6′	4″		
	2*f*.	3′	5″		
		3′	1″	7‴	8⁗
	1*f*.	10′	7″	0	
	15*f*.	0′	8″		
Ans.	17*f*.	2′	4″	7‴	8⁗

3. What is the product of 7*f*. 8′ by 6*f*. 4′ 3″?

Ans. 48*f*. 8′ 7″.

4. What is the product of 6*f*. 9′ 7″ by 4*f*. 2′?

Ans. 28*f*. 3′ 11″ 2‴.

5. What is the area of a marble slab, whose length is 7*f*. 3′, and breadth 2*f*. 11′?

6. How many square feet are contained in the floor of a hall 37*f.* 3′ long, by 10*f.* 7′ wide?

7. How many square feet are contained in a garden 100*f.* 6′ in length by 39*f.* 7′ in width?

8. How many yards of carpeting, one yard in width, will it require to cover a room 16*f.* 5′ by 13*f.* 7′?

9. What will the plastering of a ceiling cost at 13cts. a square yard, its length being 30*f.* 7 inches, and the breadth 22*f.* 4 inches?

10. How many cubic feet are contained in a rectangular stone, 7*f.* 4′ long, 2*f.* 11′ wide, and 1*f.* 10′ thick?

NEW METHOD OF MULTIPLYING DUODECIMALS.

84. Duodecimals may be multiplied together by a process similar to that employed for decimals, by the help of two new characters or figures.

For this purpose, we will represent 10 by the symbol x, and 11 by ǁ.

Examples.

1. What is the product of 3*f.* 2′ 3″ by 2*f.* 1′ 4″?

OPERATION.

```
   3.23
   2.14
  10 90
  32 3
 646
Ans. 6.89 00 =6f. 8′ 9″.
```

EXPLANATION.

In this example, we separated the feet from the parts of feet by a point, as in decimals, we then multiplied as in whole numbers, observing to carry one for every 12, when multiplying as well as when adding.

The work of the same example, by the usual rule, is as follows:

$$\begin{array}{rrrrr}
 & 3f. & 2' & 3'' & \\
 & 2f. & 1' & 4'' & \\
\hline
 & 1' & 0'' & 9''' & 0'''' \\
 & 3' & 2'' & 3''' & \\
6f. & 4' & 6'' & & \\
\hline
\text{Ans. } 6f. & 8' & 9'' & 0''' & 0'''' .
\end{array}$$

2. What is the product of $38f.\ 3'\ 1''$ by $31f.\ 2'\ 2''$?

OPERATION.

```
     32.31
     27.22
     64 62
    646 2
   1x397
   6462
```

Ans. $834.x5\ 82 = (8\times144+3\times12+4)f.\ 10'\ 5''\ 8'''\ 2''''$
$= 1092f.\ 10'\ 5''\ 8'''\ 2''''$.

EXPLANATION.

In this example, we converted $38f.$ and $31f.$, which are now in the decimal scale of notation, into the duodecimal scale of notation. In this way we found that $38=3\times12+2$; and $31=2\times12+7$, so that 38 and 31, when expressed duodecimally, become 32 and 27.

In the product the whole part, or feet, 834, is expressed

duodecimally; it is therefore equal to $8\times144+3\times12+4=$ $1192f.$, as given in the answer.

3. What is the product of $5f.\ 2'\ 3''\ 4'''$ by $2f.\ 1'\ 2''\ 5'''$?

OPERATION.

```
    5.234
    2.125
   21 ||48
   x4 68
  523 4
 x468
Ans. x.x95 x08=10f. 10' 9'' 5''' 10'''' 0''''' 8''''''
```

4. Multiply $47f.\ 3'\ 4''\ 3'''$ by $80f.\ 2'\ 3''\ 5'''$, and exhibit the work by both methods.

FIRST METHOD.

			47f.	3'	4''	3'''		
			80f.	2	3	5		
		1'	7''	8'''	4''''	9'''''	3''''''	
		11'	9''	10'''	0''''	9'''''		
	7f.	10'	6''	8'''	6''''			
	3782f.	4'	4''	0'''				
Ans.	3791f.	4'	4''	2'''	11''''	6'''''	3''''''	

SECOND METHOD.

```
       3||.343
       68.235
      178 493
      ||9x 09
     7x68 6
   2762x0
  1||7816
```

Ans. $223||.442\ ||63=(2\times1728+2\times144+3\times12+11)\,f.$ $4'\ 4''\ 2'''\ 11''''\ 6'''''\ 3''''''=3791f.\ 4'\ 4''\ 2'''\ 11''''\ 6'''''\ 3''''''$.

CHAPTER XIII.

ALLIGATION.

85. Alligation teaches the method of finding the mean value of a mixture composed of several ingredients of different values. It is usually divided into two distinct parts, viz :—*Alligation Medial*, and *Alligation Alternate*.

ALLIGATION MEDIAL.

86. Alligation Medial, teaches the method of finding the mean value of a compound, when its several ingredients and their respective values are given.

Suppose a grocer mixes 140 pounds of tea, which is worth 8*s*. per pound; 200 pounds, worth 6*s*. per pound; and 160 pounds, worth 10*s*. per pound. What is a pound of the mixture worth?

140 pounds of tea, at 8*s*. per pound, is worth 1120*s*.; 200 pounds, at 6*s*., is worth 1200*s*.; 160 pounds, at 10*s*., is worth 1600*s*. Therefore, the mixture, which is 500 pounds, is worth $1120+1200+1600=3920s$. Hence, one pound of the mixture must be worth $\frac{3920}{500}=7\frac{21}{25}s$.

Hence, to find the mean value of a compound, composed of several ingredients of different values, we have this

RULE.

Divide the sum of the values of all the ingredients by the sum of the ingredients.

Examples.

1. A wine-merchant mixed several sorts of wine, viz: 32 gallons, at 40 cents per gallon; 15 gallons, at 60 cents per gallon; 45 gallons, at 48 cents per gallon; and 8 gallons, at 85 cents per gallon. What is the value of a gallon of the mixture?

32	gallons,	at 40	cents	=	$12.80
15	"	" 60	"	=	9.00
45	"	" 48	"	=	21.60
8	"	" 85	"	=	6.80
100	gallons of mixture			=	$50.20

Therefore, one gallon of the mixture is worth $50.20 ÷ 100 = $0.502 = 50 cents and 2 mills.

2. A farmer mixes together 7 bushels of rye, worth 72 cts. per bushel; 15 bushels of corn, worth 60 cts. per bushel; and 12 bushels of wheat, worth $1.20 per bushel. What is the value of a bushel of the mixture?

Ans. 0.83\frac{11}{17}$.

3. A goldsmith melts together 11 ounces of gold, 23 carats fine; 8 ounces, 21 carats fine; 10 ounces of pure gold, and 2 pounds of alloy. How many carats fine is the mixture?

Ans. 12$\frac{25}{53}$.

It will be understood that a *carat* is a 24th part. Thus, 21 carats fine is the same as $\frac{21}{24}$ pure metal; in the same way, 23 carats fine is $\frac{23}{24}$ pure metal.

4. On a certain day, the mercury in the thermometer was observed to stand 2 hours at 62 degrees, 4 hours at 70 de-

grees, 5 hours at 72 degrees, 3 hours at 59 degrees, and 1 hour at 75 degrees. What was the mean temperature for the 15 hours?

Ans. $67\frac{11}{15}$ degrees.

5. Suppose a ship sails at the rate of 5 knots for 3 hours, at 7 knots for 5 hours, and 8 knots for 4 hours. What is her rate of sailing during the 12 hours?

6. A grocer mixes 30 pounds of sugar, worth 10 cents per pound; 40 pounds, worth $10\frac{1}{2}$ cents; 24 pounds, worth 11 cents per pound; and 60 pounds, worth 13 cents per pound. What is a pound of the mixture worth?

ALLIGATION ALTERNATE.

87. Alligation Alternate is the reverse of Alligation Medial; that is, it teaches the method of finding the ingredients, when their rates are given, so that the compound shall have a given value.

Suppose we wished to mix teas, which are worth 4 and 6 shillings per pound, so that the mixture may be worth 5 shillings per pound, it is obvious that we must take equal quantities of each; since the price of the one, is as much less than the mean price, as the other is greater.

Again, suppose we wish to mix teas, which are worth 4 and 7 shillings per pound, so that the mixture may be worth 5 shillings. In this case the 7 shilling tea is 2 shillings above the average price, whilst the 4 shilling tea is but 1 shilling below; it will be necessary to use twice as much of the 4 shilling tea as of the 7 shilling tea; and in all cases it is obvious that the quantities to be used will be in the inverse ratio to the differences between their prices and the mean price. When

there are more than two simples they may be compared together in couplets, one term of which must exceed the average price, whilst the other must be less.

CASE I.

The rates of the several ingredients being given, to make a compound of a fixed rate.

From what has been said above, we draw the following

RULE.

I. Write the rates of the simples in a line under each other, then connect each rate of the ingredients, which is less than the rate of the compound, with one or more rates greater than the rate of the compound; connect in the same way, each rate which is greater than the rate of the compound, with one or more rates which are less.

II. Write the difference between each rate of the ingredients, and the compound rate, opposite the rate of the ingredient with which it is connected. If only one difference stand against any rate, it will be the required quantity of the ingredient of that rate; but, if there be several, their sum will be the quantity required.

Examples.

1. How much sugar at 5, 6, and 10 cents per pound, must be mixed together, so that a pound of the mixture may be worth 8 cents?

SOLUTION.

$$8\left\{\begin{array}{l} 5 \\ 6 \\ 10 \end{array}\right. \qquad \begin{array}{r} 2 \\ 2 \\ 3+2=5 \end{array}$$

Therefore, if we take 2 pounds at 5 cents, 2 pounds at 6 cents, and 5 pounds at 10 cents, we shall satisfy the condi-

tions of the question. It is obvious that any other number of pounds which are to each other as the numbers 2, 2, and 5, will satisfy the question equally well; so that in Alligation the number of solutions are *indefinite*; all that we can do is to find the ratios of the quantities required.

NOTE.—In many cases the ingredients will admit of being connected in several ways, and then we shall obtain as many sets of ratios as there are methods of connecting them.

2. How many pounds of raisins at 4, 6, 8, and 10 cents per pound must be mixed, so that a pound of the compound may be worth 7 cents?

In this question the terms may be connected in seven distinct ways; therefore we shall obtain seven sets of ratios, as follows:

7	4	3
	6	1
	8	1
	10	3

7	4	1
	6	3
	8	3
	10	1

7	4	1+3=4
	6	1
	8	3+1=4
	10	3

7	4	3
	6	1+3=4
	8	1
	10	3+1=4

7	4	1
	6	1+3=4
	8	3+1=4
	10	1

NOTE.—The two remaining methods of connecting the terms, have been omitted on account of the difficulty in arranging the type in imitation of the operation.

In one of these cases, the 4 is connected with the 8 and 10; the 6 with 10, and the ratios are found to be 4, 3, 3, and 4. In the other case the 4 is connected with the 8 and 10; the 6 with 8, and 10, and the ratios are all equal.

3. How much wine, at 72 cents per gallon, and 48 cents

per gallon, must be mixed together, that the composition may be worth 60 cents per gallon?

Ans. An equal quantity of each.

4. Suppose ten pounds of pure gold, when immersed in water, to displace 4 pounds of water; 10 pounds of pure silver to displace 7 pounds of water; and 10 pounds of an alloy of gold and silver to displace 6 pounds of water. What are the proportions of gold and silver in the compound?

Ans. Twice as much silver as gold.

5. How many gallons of wine and water must be mixed together, so that the mixture shall be worth 60 cents per gallon, the water being considered of no value, and the wine with which it is mixed being worth 90 cents per gallon?

Ans. 2 gallons of wine to 1 of water.

6. Having gold of 12, 16, 17, and 22 carats fine, what proportion of each kind must I take, to make a compound of 18 carats fine?

7. It is required to mix different sorts of grain, at 56, 62, and 75 cents per bushel, so that the mixture may be worth 60 cents per bushel. How much of each kind must be taken?

8. Hiero, king of Sicily, ordered a crown to be made containing 63oz. of pure gold; but suspecting that the workmen had debased it by using part silver, he recommended the detection of the fraud to the famous Archimedes, who putting it into water found that it displaced 8.2245 cubic inches of water. He next found that a cubic inch of gold weighed 10.36 ounces, and a cubic inch of silver weighed 5.85 ounces; then, from this data, he calculated the proportions of gold and silver of which the crown was composed. What must have been his result?

Ans. 2.5447 oz. of gold to every 2.1434 oz. of silver.

CASE II.

When one of the ingredients is limited to a certain quantity, we have this

RULE.

Find the proportionate quantities of each ingredient, by Case I., in the same manner as though there was no limitation; then as the difference against the simple whose quantity is given, is to each of the other differences, so is the given quantity of that simple to the quantity required of each of the other simples.

Examples.

1. A person wishes to mix 10 bushels of wheat, worth $1 per bushel, with rye, worth 70 cents per bushel, and oats worth 30 cents per bushel, so that the mixture may be worth 60 cents per bushel. How many bushels of rye and oats must he use?

Proceeding according to Case I., we find the proportionate numbers to be 30, 30, and 50. Hence,

$$30 : 30 :: 10 : 10$$
$$30 : 50 :: 10 : 16\tfrac{2}{3}$$

So that he must make use of 10 bushels of rye, and $16\frac{2}{3}$ bushels of oats.

2. A grocer has 90 pounds of tea, worth 90 cents per pound, which he wishes to mix with three other qualities, valued at 80 cents, 70 cents, and 60 cents per pound. How much must he take of these three kinds so as to be able to sell the mixture at 85 cents per pound? Ans. 10 pounds of each.

3. A wine-merchant wishes to mix 100 gallons of wine, worth $1 per gallon, with wines worth $1.10, $1.20, and $1.30 per gallon, so that the mixture shall be worth $1.15 per gallon. How many gallons must he use?

NOTE.—This question, like question 2, under Case I., will admit of seven answers, arising from the seven ways in which the prices of the ingredients may be connected.

4. How much gold at 16, 20, and 24 carats fine, and how much alloy, must be mixed with 10 ounces of 18 carats fine, that the composition may be 22 carats fine?

5. A merchant has 90 pounds of spice worth 86 cents per pound, which he wishes to mix with three other sorts which are worth 30, 40, and 50 cents per pound, respectively. How many pounds must be used so that the compound may be worth 55 cents per pound?

CASE III.

When two or more of the ingredients are limited in quantity, we have the following

RULE.

Find, as in Alligation Medial, what will be the rate of a mixture made of the given quantities of the limited ingredients only; then consider this as the rate of a limited ingredient, whose quantity is the sum of the quantities of the limited ingredients, from which, and the rates of the unlimited ingredients, proceed to calculate the several quantities required, as in Case II.

Examples.

1. I have 36 gallons of wine, at 24 cents a gallon, 8 gallons at 52 cents, and 4 gallons at 88 cents, and would mix the whole with three other kinds of wine, one at $1.25, one at 86 cents, and the other at 90 cents per gallon. How many gallons of the last sorts must I use, so that the mixture may be worth $1 per gallon?

By Alligation Medial, we find as follows:

36 gallons	at	24	cents	=\$8.64
8 "	"	52	"	= 4.16
4 "	"	88	"	= 3.52
48 gallons come to				\$16.32

Therefore, one gallon of this mixture is worth 34 cents.

Now by Case II., of Alligation Alternate, we have

$$100\left\{\begin{array}{l}34\\125\\86\\90\end{array}\right. \qquad \begin{array}{r}25\\66+14+10=90\\25\\25\end{array}$$

$$\begin{array}{l}25 : 90 :: 48 : 172\frac{4}{5}\\25 : 25 :: 48 : 48\\25 : 25 :: 48 : 48\end{array}$$

Therefore, I must take 48 gallons of the two sorts, which are worth 86 and 90 cents per gallon, and $172\frac{4}{5}$ gallons of the sort which is worth \$1.25 per gallon.

2. With 63 gallons of wine, worth 8 shillings per gallon, I mixed other wine at 7 shillings, and some water: I then found that it was worth 6 shillings per gallon. How much wine and water did I mix with the 63 gallons?

3. A person wishes to mix 10 bushels of wheat, at 70 cents per bushel, with rye at 48 cents, corn at 36 cents, and barley at 30 cents per bushel, so that a bushel of the mixture may be worth 38 cents. What quantity of each must be taken?

4. With 50 pounds of tea, worth 75 cents per pound, I wish to mix other teas worth 90 and 95 cents per pound, so as to be able to afford the mixture at 80 cents per pound. How many pounds of the 90 and 95 cent teas must I use?

CASE IV.

When the whole compound is limited to a certain quantity, we have this

RULE.

Find the proportional parts, as in Case I.; then as the sum of the proportional parts thus obtained, is to the given quantity, so is the proportionate quantity of each ingredient, to its required quantity.

Examples.

1. Having three sorts of raisins at 9, 12, and 18 cents per pound, what quantity of each sort must I take to fill a cask of 210 pounds, so that its contents may be worth 14 cents per pound?

SOLUTION.

$$14\left\{\begin{array}{l}9\\12\\18\end{array}\right.\qquad\begin{array}{r}4\\4\\5+2=7\end{array}$$

$$\begin{array}{rcrcrcr}15 & : & 210 & :: & 4 & : & 56\\15 & : & 210 & :: & 4 & : & 56\\15 & : & 210 & :: & 7 & : & 98\\ & & & & & & \overline{210}\end{array}$$

Therefore, I must take 56 pounds each, at 9 and 12 cents per pound, and 98 pounds at 18 cents per pound.

2. A goldsmith has gold of 14, 18, and 20 carats fine; and would mix of all these sorts so much as to make a mass of 50 ounces, which shall be 16 carats fine. How much of each sort is required?

Ans. $\left\{\right.$ 30 oz. at 14 carats fine; and 10 oz. each, at 18 and 20 carats fine.

3. How much water must be mixed with brandy, worth $1.60 per gallon, to reduce the price to $1.20 per gallon, provided I fill a cask of 120 gallons?

4. A grocer has currants at 6, 8, 10, and 12 cents per pound; and he would make a mixture of 840 pounds, so that a pound of it may be worth 9 cents. How many pounds of each sort may be taken?

315	pounds, at	6	cents.	105	pounds, at	6	cents.
105	" "	8	"	315	" "	8	"
105	" "	10	"	315	" "	10	"
315	" "	12	"	105	" "	12	"
280	pounds, at	6	cents.	210	pounds at	6	cents.
70	" "	8	"	280	" "	8	"
280	" "	10	"	70	" "	10	"
210	" "	12	"	280	" "	12	"
84	pounds, at	6	cents.	240	pounds at	6	cents.
336	" "	8	"	180	" "	8	"
336	" "	10	"	180	" "	10	"
84	" "	12	"	240	" "	12	"

210	pounds, at	6	cents.
210	" "	8	"
210	" "	10	"
210	" "	12	"

In this case, the seven sets of answers arise from the seven ways in which the ratios are capable of being alligated. We may add, that when there are two rates less than the *mean*, and two greater than the *mean*, there must be seven ways in which they may be alligated, and consequently they will give rise to seven distinct sets of answers.

Questions in Alligation belong properly to that branch of algebra called *Indeterminate Analysis*, and will, in many ca-

ses, admit of an indefinite number of answers; indeed, we may combine these seven sets of answers in as many ways as we wish, so that each combination shall produce a new answer. Thus, taking $\frac{3}{5}$ of the first set, and adding them to $\frac{2}{5}$ of the second set, we obtain the following answers.

231	pounds,	at	6	cents.
189	"	"	8	"
189	"	"	10	"
331	"	"	12	"

If we take $\frac{1}{3}$ of each of the first three sets and add them, we get

$233\frac{1}{3}$	pounds,	at	6	cents.
$163\frac{1}{3}$	"	"	8	"
$233\frac{1}{3}$	"	"	10	"
210	"	"	12	"

Again, taking a seventh part of the sum of all the seven sets of answers, we get

$206\frac{4}{7}$	pounds,	at	6	cents.
$213\frac{3}{7}$	"	"	8	"
$213\frac{3}{7}$	"	"	10	"
$206\frac{4}{7}$	"	"	12	"

This method of combining can be varied in an infinite number of ways.

CHAPTER XIV.

PERMUTATION.

88. Permutation means the number of transpositions which can be made with any number of individual things, all being taken at a time.

Thus, by Permutation, we obtain the following results:

1) Perm. (*abc*) =

abc	*acb*
bac	*bca*
cab	*cba*

2) Perm. (*aabb*) =

aabb	*abab*
abba	*baab*
baba	*bbaa*

3) Perm. (*abcd*) =

abcd	*cabd*
abdc	*cadb*
acbd	*cbad*
acdb	*cbda*
adbc	*cdab*
adcb	*cdba*
bacd	*dabc*
badc	*dacb*
bcad	*dbac*
bcda	*dbca*
bdac	*dcab*
bdca	*dcba*

4) Perm. (*abbc*) =

abbc
abcb
acbb
babc
bacb
bbac
bbca
bcab
bcba
cabb
cbab
cbba

5) Perm. $(aabbc)$ =

aabbc	*babca*
aabcb	*bacab*
aacbb	*bacba*
ababc	*bbaac*
abacb	*bbaca*
abbac	*bbcaa*
abbca	*bcaab*
abcab	*bcaba*
abcba	*bcbaa*
acabb	*caabb*
acbab	*cabab*
acbba	*cabba*
baabc	*cbaab*
baacb	*cbaba*
babac	*cbbaa*

6) Perm. $(aaaabb)$ =

aaaabb
aaabab
aaabba
aabaab
aababa
aabbaa
abaaab
abaaba
ababaa
abbaaa
baaaab
baaaba
baabaa
babaaa
bbaaaa

7) Perm. $(aabcd)$ =

aabcd	*acabd*	*baacd*	*caabd*	*daabc*
aabdc	*acadb*	*baadc*	*caadb*	*daacb*
aacbd	*acbad*	*bacad*	*cabad*	*dabac*
aacdb	*acbda*	*bacda*	*cabda*	*dabca*
aadbc	*acdab*	*badac*	*cadab*	*dacab*
aadcb	*acdba*	*badca*	*cadba*	*dacba*
abacd	*adabc*	*bcaad*	*cbaad*	*dbaac*
abadc	*adacb*	*bcada*	*cbada*	*dbaca*
abcad	*adbac*	*bcdaa*	*cbdaa*	*dbcaa*
abcda	*adbca*	*bdaac*	*cdaab*	*dcaab*
abdac	*adcab*	*bdaca*	*cdaba*	*dcaba*
abdca	*adcba*	*bdcaa*	*cdbaa*	*dcbaa*

CASE I.

When the individual things are all different, to find the number of permutations.

From the above results, we deduce the following

RULE.

Multiply continually the series of natural numbers, 1, 2, 3, 4, &c., until we have used as many factors as there are individual things.

Examples.

1. How many permutations can be made of 6 individual things, all different? Ans. $1\times2\times3\times4\times5\times6=720$.

2. How many different ways may 10 persons sit at table? Ans. 3628800.

3. How many permutations may be made with 15 individual things?

CASE II.

When several of the individual things are of the same kind, we have this

RULE.

I. Find the number of permutations, by Case I., on the supposition that all the individual things are different.

II. Find, by the same Case, the number of permutations, which can be made of as many individual things as there are terms of any particular sort; do the same for all the terms which are repeated, then take the continued product of all the partial permutations, and divide the total number of permutations, as found by the first part of this rule, by this continued product.

Examples.

1. How many permutations can be made of the letters A A B B C C?

SOLUTION.

By Case I., we find the total number of permutations of six things to be $1\times2\times3\times4\times5\times6$. We also find the number of permutations of two things to be 1×2; but since there are

three letters which are repeated, each twice, our continued product of these partial permutations is $1\times2\times1\times2\times1\times2$. Therefore $\frac{1\times2\times3\times4\times5\times6}{1\times2\times1\times2\times1\times2}$, which, by canceling, reduces to 90, for the number of permutations sought.

2. How many numbers, of nine figures each, can be expressed by the use of four figure ones, three figure twos, and two figure threes? $\frac{1\times2\times3\times4\times5\times6\times7\times8\times9}{1\times2\times3\times4\times1\times2\times3\times1\times2}=1260$.

3. How many ways may the letters of the words HIGHER ARITHMETIC, be permutated? Ans. 72648576000.

4. In how many different ways may three maple trees, five ash trees, and two elm trees, be set out in a single row?

COMBINATION.

89. COMBINATION means the number of ways in which a certain number of things can be combined, by taking a given number at a time. When two things are combined at a time, the combinations are said to be of the *second class;* when three are taken at a time, the combinations are then of the *third class*, and so on, for higher classes.

CASE I.

When the combinations allow of a repetition of the same individual thing, we find the following results.

COMBINATIONS WITH REPETITIONS.

1) Comb. (a, b, c, d) of the second Class.
aa, ab, ac, ad, bb, bc, bd, cc, cd, dd.

2) Comb. (a, b, c, d, e) of the second Class.
aa, ab, ac, ad, ae, bb, bc, bd, be, cc, cd, ce, dd, de, ee.

3) Comb. (a, b, c)
(Third Class.)

aaa
aab
aac
abb
abc
acc
bbb
bbc
bcc
ccc

4) Comb. (a, b, c, d)
(Third Class.)

aaa	*bbb*
aab	*bbc*
aac	*bbd*
aad	*bcc*
abb	*bcd*
abc	*bdd*
abd	*ccc*
acc	*ccd*
acd	*cdd*
add	*ddd*

5) Comb. (a, b, c, d, e)
(Third Class.)

aaa	*add*	*bee*
aab	*ade*	*ccc*
aac	*aee*	*ccd*
aad	*bbb*	*cce*
aae	*bbc*	*cdd*
abb	*bbd*	*cde*
abc	*bbe*	*cee*
abd	*bcc*	*ddd*
abe	*bcd*	*dde*
acc	*bce*	*dee*
acd	*bdd*	*eee*
ace	*bde*	

6) Comb. (a, b, c)
(Fourth Class.)

aaaa
aaab
aaac
aabb
aabc
aacc
abbb
abbc
abcc
accc
bbbb
bbbc
bbcc
bccc
cccc

7) Comb. (a, b, c, d)
(Fourth Class.)

aaaa	abbd	bbcd
aaab	abcc	bbdd
aaac	abcd	bccc
aaad	abdd	bccd
aabb	accc	bcdd
aabc	accd	bddd
aabd	acdd	cccc
aacc	addd	cccd
aacd	bbbb	ccdd
aadd	bbbc	cddd
abbb	bbbd	dddd
abbc	bbcc	

8) Comb. $(a, b, c, d, e.)$
(Fourth Class.)

aaaa	abbe	bbbc	bdee
aaab	abcc	bbbd	beee
aaac	abcd	bbbe	cccc
aaad	abce	bbcc	cccd
aaae	abdd	bbcd	ccce
aabb	abde	bbce	ccdd
aabc	abee	bbdd	ccde
aabd	accc	bbde	ccee
aabe	accd	bbee	cddd
aacc	acce	bccc	cdde
aacd	acdd	bccd	cdee
aace	acde	bcce	ceee
aadd	acee	bcdd	dddd
aade	addd	bcde	ddde
aaee	adde	bcee	ddee
abbb	adee	bddd	deee
abbc	aeee	bdde	eeee
abbd	bbvb		

From an inspection of the above, we deduce the following

RULE.

Add the number denoting the class of the combinations to the number of individual things, and from the sum subtract one, multiply this remainder continually by the successive decreasing terms of the series of natural numbers, until we reach the term denoting the number of individual things; then divide this product by the number of permutations, found by Case I., Art. 88, of a number of individual things denoted by the class of the combinations.

Examples.

1. How many combinations of the 5th class can be formed out of 10 individual things?

SOLUTION.

The number denoting the class added to the number of individual things gives, $10+5=15$, from which, subtracting one, we get 14. Therefore, $\frac{14\times 13\times 12\times 11\times 10}{1\times\ 2\times\ 3\times\ 4\times\ 5}=2002$, is the number of combinations sought.

2. How many different numbers, of four places of figures each, can be formed out of the nine digits?

Ans. $\frac{12\times 11\times 10\times 9}{1\times\ 2\times\ 3\times 4}=495$.

3. How many different combinations, of six things at a time, may be formed out of 11 individual things?

Ans. 8008.

CASE II.

If we form combinations, without repetitions, we shall have as follows:

COMBINATIONS WITHOUT REPETITIONS.

1) Comb. $(a, b, c, d, e, f, g, h, i)$

(Second Class.)

ab	*ah*	*bg*	*cg*	*dh*	*fg*
ac	*ai*	*bh*	*ch*	*di*	*fh*
ad	*bc*	*bi*	*ci*	*ef*	*fi*
ae	*bd*	*cd*	*de*	*eg*	*gh*
af	*be*	*ee*	*df*	*eh*	*gi*
ag	*be*	*cf*	*dg*	*ei*	*hi*

2) Comb. (a, b, c, d, e, f)

(Third Class.)

abc	*acd*	*adf*	*bcf*	*cde*
abd	*ace*	*aef*	*bde*	*cdf*
abe	*acf*	*bcd*	*bdf*	*cef*
abf	*ade*	*bce*	*bef*	*def*

3) Comb. (a, b, c, d, e, f, g, h)

(Third Class.)

abc	*adf*	*bcf*	*bfh*	*cgh*
abd	*adg*	*bcg*	*bgh*	*def*
abe	*adh*	*bch*	*cde*	*deg*
abf	*aef*	*bde*	*cdf*	*deh*
abg	*aeg*	*bdf*	*cdg*	*dfg*
abh	*aeh*	*bdg*	*cdh*	*dfh*
acd	*afg*	*bdh*	*cef*	*dgh*
ace	*afh*	*bef*	*ceg*	*efg*
acf	*agh*	*beg*	*ceh*	*efh*
acg	*bcd*	*beh*	*cfg*	*egh*
ach	*bce*	*bfg*	*cfh*	*fgh*
ade				

4) Comb. (a, b, c, d, e, f, g)

(Fourth Class.)

abcd	*abfg*	*adfg*	*bdeg*
abce	*acde*	*aefg*	*bdfg*
abcf	*acdf*	*bcde*	*befg*
abcg	*acdg*	*bcdf*	*cdef*
abde	*acef*	*bcdg*	*cdeg*
abdf	*aceg*	*bcef*	*cdfg*
abdg	*acfg*	*bceg*	*cefg*
abef	*adef*	*bcfg*	*defg*
abeg	*adeg*	*bdef*	

5) Comb. (a, b, c, d, e, f, g, h)

(Fifth Class.)

abcde	*abdfh*	*acegh*	*bcefh*
abcdf	*abdgh*	*acfgh*	*bcegh*
abcdg	*abefg*	*adefg*	*bcfgh*
abcdh	*abefh*	*adefh*	*bdefg*
abcef	*abegh*	*adegh*	*bdefh*
abceg	*abfgh*	*adfgh*	*bdegh*
abceh	*acdef*	*aefgh*	*bdfgh*
abcfg	*acdeg*	*bcdef*	*befgh*
abcfh	*acdeh*	*bcdeg*	*cdefg*
abcgh	*acdfg*	*bcdeh*	*cdefh*
abdef	*acdfh*	*bcdfg*	*cdegh*
abdeg	*acdgh*	*bcdfh*	*cdfgh*
abdeh	*acefg*	*bcdgh*	*cefgh*
abdfg	*acefh*	*bcefg*	*defgh*

Therefore, when there are no repetitions in the combinations, we have this

RULE.

Add one to the number of individual things, and from the sum subtract the number denoting the class of the combinations, multiply this remainder by the successive increasing terms of the series of natural numbers, till we reach the term denoting the number of individual things; then divide this product by the number of permutations, found by Case I., Art. 88, of a number of individual things, denoted by the class of the combinations.

Examples.

1. What is the number of combinations, without repetitions, of the fourth class, which can be formed out of 10 individual things?

SOLUTION.

Adding one to the number of individual things, we get 11, from which, subtracting the number denoting the class, we get $11-4=7$. Therefore, $\frac{7\times8\times9\times10}{1\times2\times3\times4}=210$, is the number sought.

2. How many lottery tickets, each having three numbers, can be formed out of 60 numbers?

Ans. $\frac{58\times59\times60}{1\times2\times3}=34220$.

3. How many combinations, of 4 things at a time, without repetitions, can be formed out of 100 individual things?

VARIATION.

90. By VARIATION, we understand the different transpo-

sitions that can take place when the individual things are not all taken at once. Variations are divided into classes, in the same way as Combinations. They are also distinguished by variations with repetitions, and variations without repetitions.

CASE I.

When the variations are made with repetitions, we have as follows:

VARIATIONS WITH REPETITIONS.

1) Var. (a, b, c, d, e, f)
(Second Class.)

aa	*ba*	*ca*	*da*	*ea*	*fa*
ab	*bb*	*cb*	*db*	*eb*	*fb*
ac	*bc*	*cc*	*dc*	*ec*	*fc*
ad	*bd*	*cd*	*dd*	*ed*	*fd*
ae	*be*	*ce*	*de*	*ee*	*fe*
af	*bf*	*cf*	*df*	*ef*	*ff*

2) Var. (a, b, c, d)
(Third Class.)

aaa	*adb*	*bcc*	*cbd*	*dba*
aab	*adc*	*bcd*	*cca*	*dbb*
aac	*add*	*bda*	*ccb*	*dbc*
aad	*baa*	*bdb*	*ccc*	*dbd*
aba	*bab*	*bdc*	*ccd*	*dca*
abb	*bac*	*bdd*	*cda*	*dcb*
abc	*bad*	*caa*	*cdb*	*dcc*
abd	*bba*	*cab*	*cdc*	*dcd*
aca	*bbb*	*cac*	*cdd*	*dda*
acb	*bbc*	*cad*	*daa*	*ddb*
acc	*bbd*	*cba*	*dab*	*ddc*
acd	*bca*	*cbb*	*dac*	*ddd*
ada	*bcb*	*cbc*	*dad*	

3) Var. (a, b, c)

(Fourth Class.)

aaaa	*abcc*	*baca*	*bcbb*	*cbac*
aaab	*acaa*	*bacb*	*bcbc*	*cbba*
aaac	*acab*	*bacc*	*bcca*	*cbbb*
aaba	*acac*	*bbaa*	*bccb*	*cbbc*
aabb	*acba*	*bbab*	*bccc*	*cbca*
aabc	*acbb*	*bbac*	*caaa*	*cbcb*
aaca	*acbc*	*bbba*	*caab*	*cbcc*
aacb	*acca*	*bbbb*	*caac*	*ccaa*
aacc	*accb*	*bbbc*	*caba*	*ccab*
abaa	*accc*	*bbca*	*cabb*	*ccac*
abab	*baaa*	*bbcb*	*cabc*	*ccba*
abac	*baab*	*bbcc*	*caca*	*ccbb*
abba	*baac*	*bcaa*	*cacb*	*ccbc*
abbb	*baba*	*bcab*	*cacc*	*ccca*
abbc	*babb*	*bcac*	*cbaa*	*cccb*
abca	*babc*	*bcba*	*cbab*	*cccc*
abcb				

From the above, we draw the following

RULE.

Raise the number denoting the individual things to a power whose exponent is the number expressing the class of variations.

Examples.

1. How many variations, with repetitions, can be formed of the 4th class, out of 5 individual things?

Ans. $5^4 = 625$.

2. How many numbers can we form out of the nine digits, of 9 places of figures each, provided we are allowed to make a repetition of the digits? Ans. $9^9 = 387420489$.

CASE II.

When the variations are formed, without repetitions, we find as follows:

VARIATIONS WITHOUT REPETITIONS.

Var. (a, b, c, d, e)

(Fourth Class.)

abcd	*aecb*	*bdea*	*cdab*	*dbca*	*eadb*
abce	*aecd*	*bdec*	*cdae*	*dbce*	*eadc*
abdc	*aedb*	*beac*	*cdba*	*dbea*	*ebac*
abde	*aedc*	*bead*	*cdbe*	*dbec*	*ebad*
abec	*bacd*	*beca*	*cdea*	*dcab*	*ebca*
abed	*bace*	*becd*	*cdeb*	*dcae*	*ebcd*
acbd	*badc*	*beda*	*ceab*	*dcba*	*ebda*
acbe	*bade*	*bedc*	*cead*	*dcbe*	*ebdc*
acdb	*baec*	*cabd*	*ceba*	*dcea*	*ecab*
acde	*baed*	*cabe*	*cebd*	*dceb*	*ecad*
aceb	*bcad*	*cadb*	*ceda*	*deab*	*ecba*
aced	*bcae*	*cade*	*cedb*	*deac*	*ecbd*
adbc	*bcda*	*caeb*	*dabc*	*deba*	*ecda*
adbe	*bcde*	*caed*	*dabe*	*debc*	*ecdb*
adcb	*bcea*	*cbad*	*dacb*	*deca*	*edab*
adce	*bced*	*cbae*	*dace*	*decb*	*edac*
adeb	*bdac*	*cbda*	*daeb*	*eabc*	*edba*
adec	*bdae*	*cbde*	*daec*	*eabd*	*edbc*
aebc	*bdca*	*cbea*	*dbac*	*eacb*	*edca*
aebd	*bdce*	*cbed*	*dbae*	*eacd*	*edcb*

From the above, we deduce the following

RULE.

Add one to the number of individual things, and from the sum subtract the number denoting the order of the class of variations; multiply this remainder continually by the successive increasing terms of the series of natural numbers, until we reach that term which is equal to the number of individual things.

Examples.

1. How many variations, without repetitions, of the 4th class, can be formed out of 10 individual things?

Ans. $7 \times 8 \times 9 \times 10 = 5040$.

2. How many numbers, of five places of figures each, can be formed out of the 9 digits, so that the same digit shall not be repeated in the same number?

Ans. $5 \times 6 \times 7 \times 8 \times 9 = 15120$.

3. How many words, of 13 letters each—no letter being repeated—can be formed out of the 26 letters of the alphabet, on the supposition that every combination of letters is capable of producing a word?

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91. It is required to find the number of ways in which an odd number of letters, arranged as in the following squares, can be read in alphabetical order, by beginning at the central square, and reading outwards.

1

c	b	c
b	a	b
c	b	c

2

e	d	c	d	e
d	c	b	c	d
c	b	a	b	c
d	c	b	c	d
e	d	c	d	e

3

g	f	e	d	e	f	g
f	e	d	c	d	e	f
e	d	c	b	c	d	e
d	c	b	a	b	c	d
e	d	c	b	c	d	e
f	e	d	c	d	e	f
g	f	e	d	e	f	g

By carefully inspecting the above figures, we derive the following

RULE.

Find, by Case II., Art. 89, the number of combinations, without repetitions, of a number of individual things, which is one less than the number of letters, the class of the combination being represented by half the number of letters after one is subtracted: then multiply this result by 4.

Examples.

1. How many ways may the first nine letters of the alphabet be read in this way?

SOLUTION.

Subtracting one from the number of letters, we get $9-1=8$; therefore, by Case II., Art. 89, we find the number of combinations of 8 things, without repetitions, of the fourth class, to be $\frac{8\times7\times6\times5}{4\times3\times2\times1}=70$; this multiplied by 4, gives 280 for the number sought.

2. In how many ways may the phrase, MODERATE YOUR CURIOSITY, be read after the above manner?

Ans. 739024.

3. In how many ways may the 26 letters of the alphabet be read in this way?

CHAPTER XV.

MISCELLANEOUS QUESTIONS SOLVED BY ANALYSIS.

92. A man and his wife usually drank out a cask of beer in 12 days; but when the man was from home, it lasted the woman 30 days. How many days would the man alone be in drinking it?

SOLUTION.

Since it requires 12 days for the man and his wife to drink out the cask, they must, in each day, drink $\frac{1}{12}$ of it.

Again, since the woman is 30 days in drinking it, she must, in each day, drink $\frac{1}{30}$ of it.

Hence, the fractional part which the man drank in one day must be $\frac{1}{12}-\frac{1}{30}=\frac{1}{20}$; $\therefore$ in 20 days he could drink the whole.

2. A person bought several gallons of wine for \$94, and after using 7 gallons himself, sold $\frac{1}{4}$ of the remainder for \$20. How many gallons had he at first?

SOLUTION.

Since he sold $\frac{1}{4}$ of the remainder, after using 7 gallons, for \$20, he could have sold the whole of the remainder for \$80: therefore, the value of the 7 gallons, which he used, was $94-80=\$14$; and one gallon must have cost $\frac{14}{7}=\$2$. The wine being worth \$2 per gallon, he must have purchased $\frac{94}{2}=47$ gallons.

3. A person in play, lost $\frac{1}{4}$ of his money, and then won 3 shillings, after which, he lost $\frac{1}{3}$ of what he then had; and this done, he found that he had but 12 shillings remaining. How much had he at first?

SOLUTION.

Since, in the first place, he lost $\frac{1}{4}$ of his money, he must have had left $\frac{3}{4}$ of it, to which, adding the 3 shillings which he won, he had $\frac{3}{4}$ of his money + 3 shillings. Again, he loses $\frac{1}{3}$ of this, and consequently retains $\frac{2}{3}$ of it; so that $\frac{2}{3}$ of $\frac{3}{4}$ of his money + $\frac{2}{3}$ of 3 shillings, or which is the same, $\frac{1}{2}$ of his money + 2 shillings, is what he finally had remaining; this, by the conditions of the question, is 12 shillings, ∴ we have this relation: $\frac{1}{2}$ of his money + 2 shillings, must equal 12 shillings, or which is the same, $\frac{1}{2}$ of his money must equal 10 shillings; and consequently his money must have been 20 shillings.

4. A fish was caught whose tail weighed 9 pounds, his head weighed as much as his tail and half his body, and his body weighed as much as his head and tail together. What was the weight of the fish?

SOLUTION.

Since the head of the fish is equal to $\frac{1}{2}$ of the body, together with the tail = 9 pounds, it follows, that the head and tail together must equal $\frac{1}{2}$ of the body + 18 pounds. But by the question, the head and tail together is equal to the whole body; ∴ we have this relation: $\frac{1}{2}$ of the body + 18 pounds, must equal the whole body; consequently, $\frac{1}{2}$ of the body must equal 18 pounds, and the whole body is 36 pounds. And since the body weighed as much as the head and tail together, it follows, that the weight of the whole fish was twice that of the body, or eight times that of the tail; which is 72 pounds.

5. A person engaged a workman for 48 days. For each day that he labored he received 24 cents, and for each day that he was idle, he paid 12 cents for his board. At the end of the 48 days, the account was settled, when the laborer received $5.04. Required the number of working days, and the number of days he was idle.

SOLUTION.

Had he worked all the time he would have received $24\times 48=\$11.52$; but he received only $5.04. Therefore by being idle he lost $\$11.52-\$5.04=\$6.48$. Now for each idle day, he loses the 24 cents, which he might have earned, as well as the 12 cents which he gives for his board; so that every idle day is to him a loss of $24+12=36$ cents. But we have just shown that his total loss was $6.48; $\therefore$ the number of idle days was $\frac{648}{36}=18$; and he worked $48-18=30$ days.

6. A gentleman bought two pieces of silk, which together measured 36 yards. Each of them cost as many shillings per yard as there were yards in the piece, and their whole prices were as 4 to 1. What were the lengths of the pieces?

SOLUTION.

Since each piece cost, per yard, as many shillings as there were yards in its length, it follows that their values expressed in shillings must be as the squares of their lengths. By the question, their prices were as 4 to 1; therefore the squares of their lengths must be to each other as 4 to 1; consequently, their lengths must be to each other as 2 to 1.

The question is now reduced to the following: Divide 36 into two parts, which shall be to each other as 2 to 1. These parts are $\frac{2}{3}$ of $36=24$, and $\frac{1}{3}$ of $36=12$.

7. In a mixture of wine and cider, $\frac{1}{2}$ of the whole, plus 25 gallons, is wine, and $\frac{1}{3}$ of the whole, minus 5 gallons, is cider. How many gallons were there of each?

SOLUTION.

By the question, the wine$=\frac{1}{2}$ of the whole$+25$ gallons, and the cider $=\frac{1}{3}$ of the whole $-$ 5 gallons. Hence, taking the sum of these expressions, we get the whole$=(\frac{1}{2}+\frac{1}{3})$ or $\frac{5}{6}$ of the whole $+$ 20 gallons ; $\therefore$ $\frac{1}{6}$ of the whole equals 20 gallons ; consequently, the whole is 120 gallons.

Now $\frac{1}{2}$ of the whole is 60 gallons, to which, add 25 gallons, we get for the wine, 85 gallons.

Again, $\frac{1}{3}$ of the whole is 40 gallons, from which, subtracting 5 gallons, we get for the cider, 35 gallons.

8. A market woman bought a certain number of eggs, at 2 for a penny ; and as many more, at 3 for a penny ; and having sold them again, altogether, at the rate of 5 for 2 pence, found that she had lost 4 pence. How many eggs had she?

SOLUTION.

Since, by the question, half of the eggs cost $\frac{1}{2}$ of a penny a piece, and the other half cost $\frac{1}{3}$ of a penny a piece, it follows that the average price, which she gave for the eggs, was $(\frac{1}{2}+\frac{1}{3})\div 2=\frac{5}{12}$ of a penny a piece. Since she sold them altogether at the rate of 5 for 2 pence, that is, $\frac{2}{5}$ of a penny a piece, she must have lost, on each egg, $\frac{5}{12}-\frac{2}{5}=\frac{1}{60}$ of a penny. Therefore, to lose 1 penny, she must dispose of 60 eggs ; and to lose 4 pence, she must have had 240 eggs.

9. A and B can, together, do a piece of work in 8 days ; A and C can, together, do it in 9 days. How many days would it require for each to perform the work alone?

SOLUTION.

Since A and B can do the work in 8 days, they can, in one day, do $\frac{1}{8}$ part of it; for a similar reason, A and C can do $\frac{1}{9}$ part of it in one day; B and C can do $\frac{1}{10}$ part of it in one day. Adding these fractional parts together, and observing that each individual has been included twice, we shall get, by dividing the sum by 2, the following fraction: $(\frac{1}{8}+\frac{1}{9}+\frac{1}{10})\div 2=\frac{121}{720}$, which is the fractional part of the work, which they all together would perform in one day.

We have already seen, that the part which B and C can perform in one day is $\frac{1}{10}$; $\therefore$ $\frac{121}{720}-\frac{1}{10}=\frac{49}{720}$ is the fractional part which A could perform in one day. Hence, the time in which A could alone perform the work is $\frac{720}{49}=14\frac{34}{49}$ days.

Again, the fractional part, which A and C together could perform, is $\frac{1}{9}$; $\therefore$ $\frac{121}{720}-\frac{1}{9}=\frac{41}{720}$, is the fractional part which B could perform in one day; hence, the time in which B could alone perform the work is $\frac{720}{41}=17\frac{23}{41}$ days. The fractional part which A and B together could perform, is $\frac{1}{8}$; $\therefore$ $\frac{121}{720}-\frac{1}{8}=\frac{31}{720}$ is the fractional part which C could perform in one day; hence, the time in which he could alone perform the work is $\frac{720}{31}=23\frac{7}{31}$ days.

10. A and B have each the same income. A contracts an annual debt amounting to $\frac{1}{7}$ of it; B lives on $\frac{4}{5}$ of it; and at the end of ten years B lends to A enough to pay off his debts, and has $160 left. What is the income?

SOLUTION.

Since B lives on $\frac{4}{5}$ of his income, he must save $\frac{1}{5}$ of it. A's debt for one year being $\frac{1}{7}$ of the income, B will have left after paying A's debt, $\frac{1}{5}-\frac{1}{7}=\frac{2}{35}$ of his income. And since this would in ten years amount to $160, $\frac{2}{35}$ of his income must equal $160. Hence, the incomo was $\frac{35}{2}$ of $16=$280.

11. **A merchant supported himself 3 years for $50 a year; at the end of each year he added to that part of his stock which was not thus expended, a sum equal to one-third of this part. At the end of the third year his original stock was doubled. What was the stock?**

SOLUTION.

After supporting himself the first year, he will have his original stock—$50; this, increased by its third part, will become $\frac{4}{3}$ of his original stock—$\frac{4}{3}$ of $50; living upon another $50, he will have left $\frac{4}{3}$ of his original stock,—$\frac{4}{3}$ of $50,—$50. This must again be increased by its third part, giving $\frac{16}{9}$ of his original stock—$\frac{16}{9}$ of $50—$\frac{4}{3}$ of $50.

Again, living upon $50, he will have left $\frac{16}{9}$ of his original stock—$\frac{16}{9}$ of $50—$\frac{4}{3}$ of $50—$50; increasing this once more by its third part, we get $\frac{64}{27}$ of his original stock—$\frac{64}{27}$ of $50—$\frac{16}{9}$ of $50—$\frac{4}{3}$ of $50. This, by the question, is equal to twice his original stock, or to $\frac{54}{27}$ of his original stock.

Hence, $\frac{64}{27}-\frac{54}{27}=\frac{10}{27}$ of his original stock must equal $\frac{64}{27}$ of $50+$\frac{16}{9}$ of $50+$\frac{4}{3}$ of 50 dollars$=\frac{148}{27}$ of 50=\frac{7400}{27}$ dollars; ∴ his stock was $\frac{7400}{27}\div\frac{10}{27}=$ $740.

12. **Fourteen oxen have in three weeks eaten all the grass which grew on 2 acres of land, in such a manner that they not only ate all the grass which at first was there, but also that which grew during the time they were grazing. In like manner, have 16 oxen, in four weeks, eaten up all the grass upon 3 acres of land. How many oxen can, in this way, graze for 5 weeks, upon 6 acres of land?**

SOLUTION.

By the first condition of the question, we see that the growth of 2 acres for 3 weeks is $2\times3=6$ times the growth of one acre

for one week; $\therefore$ the quantity which 14 oxen ate in 3 weeks is 2 acres, together with 6 times the growth of one acre for one week. Hence, one ox in one week would eat $\frac{1}{14}$ of $\frac{1}{3}$ of 2 acres, together with $\frac{1}{14}$ of $\frac{1}{3}$ of 6 times the growth of one acre for one week.

Again, by the second condition of the question, we see that the growth of 3 acres for 4 weeks is $3 \times 4 = 12$ times the growth of 1 acre for 1 week; $\therefore$ the quantity which 16 oxen in 4 weeks ate, is 3 acres, together with 12 times the growth of 1 acre for 1 week. Hence, one ox, in one week, would eat $\frac{1}{16}$ of $\frac{1}{4}$ of 3 acres, together with $\frac{1}{16}$ of $\frac{1}{4}$ of 12 times the growth of one acre for one week.

Now, by the nature of the question, an ox, in the former case, must eat the same as one in the latter; therefore, $\frac{1}{14}$ of $\frac{1}{3}$ of 2 acres, together with $\frac{1}{14}$ of $\frac{1}{3}$ of 6 times the growth of one acre for one week must equal $\frac{1}{16}$ of $\frac{1}{4}$ of 3 acres, together with $\frac{1}{16}$ of $\frac{1}{4}$ of 12 times the growth of one acre for one week, or which is the same thing, $\frac{1}{21}$ of an acre $+ \frac{1}{7}$ of the growth of one acre for one week is equal to $\frac{3}{64}$ of an acre $+ \frac{3}{16}$ of the growth of one acre for one week.

Hence, $\frac{1}{21} - \frac{3}{64} = \frac{1}{1344}$ of an acre must equal $\frac{3}{16} - \frac{1}{7} = \frac{5}{112}$ of the growth of one acre for one week: therefore, the growth of one acre for one week is $\frac{1}{1344} \div \frac{5}{112} = \frac{1}{60}$ of an acre.

Now, by the third condition of the question, there are 6 acres, which increase by the growth of grass for 5 weeks; and since this increase is $\frac{1}{60}$ of an acre, for one acre for one week, it follows that 6 acres in 5 weeks will increase $6 \times 5 \times \frac{1}{60} = \frac{1}{2}$ of an acre; so that there will be $6\frac{1}{2}$ acres to be eaten in 5 weeks.

Now, we have already seen, by the first condition of the question, that one ox in one week would eat $\frac{1}{14}$ of $\frac{1}{3}$ of 2 acres

$+\frac{1}{12}$ of $\frac{1}{8}$ of 6 times the growth of one acre for one week, or which is the same thing, one ox in one week would eat $\frac{1}{21}$ of an acre $+\frac{1}{420}$ of an acre $=\frac{1}{20}$ of an acre.

Consequently, one ox would in 5 weeks eat $5\times\frac{1}{20}=\frac{1}{4}$ of an acre. Therefore, the number of oxen necessary to eat the $6\frac{1}{2}$ acres in 5 weeks, must be $\frac{13}{2}\div\frac{1}{4}=26$ oxen, the answer.

13. A, B, C, D, and E, play together on this condition: that he who loses shall give to all the rest as much as they already have. First, A loses, then B, then C, then D, and at last also E. All lose in turn, and yet at the end of the fifth game they have all the same sum, viz: each $32. How much had each before they began to play?

SOLUTION.

The solution of this question is the most readily effected by a reverse process, that is, by beginning with the last game, and playing them all in a reverse order, as follows: First, take from A, B, C, and D, half they have, and add it to E's money; second, take from A, B, C, and E, half what they now have, and add it to D's; third, take from A, B, D, and E, half what they now have, and add it to C's; fourth, take half of A's, C's, D's, and E's, and add it to B's; lastly, take half of B's, C's, D's, and E's, and add it to A's.

These successive operations may be exhibited as follows:

A	B	C	D	E	
$32	32	32	32	32	end of 5th game,
16	16	16	16	96	end of 4th game,
8	8	8	88	48	end of 3d game,
4	4	84	44	24	end of 2d game,
2	82	42	22	12	end of 1st game,
Ans. 81	41	21	11	6	before playing.

14. A father left to his three sons, whose ages are 8, 10, and 13 years, \$10000, to be so divided that the respective parts being placed out at 5 per cent., compound interest, should amount to equal sums when they became 21 years of age. What are the parts?

SOLUTION.

By the question, their respective shares would be at interest 13, 11, and 8 years.

We find, by table under Art. 70, the present worth of \$1 for 13, 11, and 8 years respectively, at 5 per cent., compound interest, to be \$0.530321, \$0.584679, and \$0.676839. Now it is obvious that the parts must be to each other in the same ratio as the numbers 530321, 584679, and 676839; the sum of these numbers is 1791839. Hence, the parts are as follows:

The first one's part is $\frac{530321}{1791839}$ of \$10000=\$2959.646.

The second one's part is $\frac{584679}{1791839}$ of \$10000=\$3263.011.

The third one's part is $\frac{676839}{1791839}$ of \$10000=\$3777.343.

15. Find what each of the four persons, A, B, C, and D, are worth, by knowing,

First, that A's money, together with $\frac{1}{3}$ of B's, C's, and D's, is equal to \$137.

Secondly, that B's money, together with $\frac{1}{4}$ of A's, C's, and D's, is equal to \$137.

Thirdly, that C's money, together with $\frac{1}{5}$ of A's, B's, and D's, is equal to \$137.

Fourthly, that D's money, together with $\frac{1}{6}$ of A's, B's, and C's, is equal to \$137.

SOLUTION.

It is evident that $\frac{1}{3}$ of B's, C's, and D's, money is the same as $\frac{1}{3}$ of the *sum of all*, MINUS $\frac{1}{3}$ of A's: therefore, A's, together with $\frac{1}{3}$ of B's, C's, and D's, is equal to A's$+\frac{1}{3}$ of the *sum of all*$-\frac{1}{3}$ of A's, which, by the first condition, equals \$137.

Consequently, $\frac{2}{3}$ of A's$=\$137-\frac{1}{3}$ of the *sum of all*.

$\therefore$ A's$=\frac{3}{2}$ of $\$137-\frac{1}{2}$ of the *sum of all*. In a similar way we get B's$=\frac{4}{3}$ of $\$137-\frac{1}{3}$ of the *sum of all*.

C's$=\frac{5}{4}$ of $\$137-\frac{1}{4}$ of the *sum of all*.

D's$=\frac{6}{5}$ of $\$137-\frac{1}{5}$ of the *sum of all*.

Taking the sum of these values, of A, B, C, and D, we get the *sum of all*$=(\frac{3}{2}+\frac{4}{3}+\frac{5}{4}+\frac{6}{5})$ of $\$137-(\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5})$ of the *sum of all*. $\therefore (1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5})$ of the *sum of all*$=(\frac{3}{2}+\frac{4}{3}+\frac{5}{4}+\frac{6}{5})$ of \$137. And the *sum of all*$=\dfrac{\frac{3}{2}+\frac{4}{3}+\frac{5}{4}+\frac{6}{5}}{1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}}$ of $\$137=\frac{317}{137}$ of $\$137=\$317$.

This value, for the *sum of all*, being substituted in the above values of A, B, C, and D, we obtain the following results:

A's$=\frac{3}{2}$ of $\$137-\frac{1}{2}$ of $\$317=\$47$
B's$=\frac{4}{3}$ of $\$137-\frac{1}{3}$ of $\$317=\$77$
C's$=\frac{5}{4}$ of $\$137-\frac{1}{4}$ of $\$317=\$92$
D's$=\frac{6}{5}$ of $\$137-\frac{1}{5}$ of $\$317=\$101$

NOTE.—It is obvious, that this method of solving the above question will apply in the case of any number of unknown quantities which are similarly related to each other.

16. A and B settling accounts, found that if £6 were added to $\frac{3}{5}$ of A's money, and the same sum taken from $\frac{3}{5}$ of B's, the *sum* would be $\frac{7}{5}$ of the *remainder*, and that the sum and remainder added together made £72. What was each man's money?

SOLUTION.

Since £6 was added to $\frac{2}{3}$ of A's money, and subtracted from $\frac{2}{3}$ of B's money, the sum of A's and B's money, after this addition and subtraction, is the same as it would have been, had no such addition and subtraction been made. Therefore $\frac{2}{3}$ of A's and B's money is, by the question, £72.

Again, by the question, $\frac{2}{3}$ of A's, increased by £6, is equal to $\frac{7}{8}$ of $\frac{2}{3}$ of B's, diminished by £6; $\therefore$ $\frac{2}{3}$ of A's, increased by £6, is to $\frac{2}{3}$ of B's, diminished by £6, as 7 to 8. But we have already seen, that $\frac{2}{3}$ of A's, increased by £6, added to $\frac{2}{3}$ of B's, diminished by £6, is £72. Hence, if we divide £72 into two parts, which are to each other as 7 to 8, these parts will be $\frac{2}{3}$ of A's, increased by £6, and $\frac{2}{3}$ of B's, diminished by £6. The parts of £72 pounds, which are to each other as 7 to 8, are $\frac{7}{15}$ of £72=£33$\frac{3}{5}$, and $\frac{8}{15}$ of £72=£38$\frac{2}{5}$. Therefore $\frac{2}{3}$ of A's, increased by £6, is equal to £33$\frac{3}{5}$, consequently, $\frac{2}{3}$ of A's is £33$\frac{3}{5}$−£6=£27$\frac{3}{5}$; and the whole of A's money is $\frac{3}{2}$ of £27$\frac{3}{5}$=£41$\frac{2}{5}$. And $\frac{2}{3}$ of B's, diminished by £6, is equal to £38$\frac{2}{5}$. Therefore, $\frac{2}{3}$ of B's is £38$\frac{2}{5}$+£6=£44$\frac{2}{5}$; and the whole of B's money is $\frac{3}{2}$ of £44$\frac{2}{5}$=£66$\frac{3}{5}$.

17. A purse of \$2850 is to be divided among three persons, A, B, and C. A's share is to be to B's, as 6 to 11, and C is to have \$300 more than A and B together. What is each one's share?

SOLUTION.

Since C is to have \$300 more than A and B together, it follows that A and B must have half of what is left, after subtracting \$300: hence, A and B together have \$1275; this divided into two parts, which are to each other as 6 to 11, gives A's=$\frac{6}{17}$ of 1275=\$450; B's=$\frac{11}{17}$ of 1275=\$825; C's is evidently \$1575.

18. Two persons, A and B, purchase in company, 100 acres of land for $1000, of which the southern portion is of rather the best quality. In the division of it, A says to B, let me have of the southern portion, and I will pay for my part $\$1\frac{1}{18}$ per acre more than you pay for yours. How much land must each have, and at what price per acre?

NOTE.—This, and like questions, can be solved by the following

RULE.

Divide half the whole cost by the whole number of acres, and to the square of the quotient add the square of half the difference of the prices per acre; then extract the square root of the sum, and to this root add the quotient of half the whole cost, divided by the whole number of acres. This last sum, increased by half the difference of the prices per acre, will give the price per acre of the best land; and diminished by the same, will give the price per acre of the poorest.

Applying the above rule, we find the quotient of half the whole cost, divided by the whole number of acres, to be 5, which squared, gives 25; this increased by the square of half the difference of the price per acre, becomes $25+\frac{361}{1296}=\frac{32761}{1296}$, whose square root is $\frac{181}{36}$; this root added to 5, gives $\frac{361}{36}=10\frac{1}{36}$. Therefore, the price per acre, of A's land, is $10\frac{1}{36}+\frac{19}{36}=\$10\frac{5}{9}$. The price of B's land is $10\frac{1}{36}-\frac{19}{36}=\$9\frac{1}{2}$.

$\$500\div\$10\frac{5}{9}=47\frac{1}{19}$ acres, for A's portion.

$\$500\div\$9\frac{1}{2}=52\frac{12}{19}$ acres, for B's portion.

19. A boy divided his apples among his four companions, in the following manner: To the first he gave half an apple more than half his whole number; to the second, he likewise gave half an apple more than half the number which he then had; in the same manner he divided with the third and fourth companion, giving to each half an apple more than half the number which was left after giving to the preceding one.

After having divided with the fourth companion, he had but one apple left. How many had he at first?

SOLUTION.

This question is most easily solved by beginning with the last companion, and reversing each successive operation.

Since he had but one apple left, after making the last division, he must have had 3 after the third division, because 3 diminished by ½ more than its half, leaves 1; for a similar reason, he must have had 7 after the second division. In this way we may retrace the process by multiplying the number by 2, and adding 1 to the product, for each successive step. In this way we find that after the first division he must have had $2\times7+1=15$. And before he divided with the first, he must have had $2\times15+1=31$ apples.

NOTE.—From the above method of solving this question, we see that it would not be difficult to extend the solution to the case of any number of divisions. Indeed, we see that the number required is always one less than 2, raised to a power whose index is one more than the number of divisions.

Thus, had there been 10 companions to divide with, after the above method, he must have had $2^{11}-1=2047$ apples at first.

This method of division is effected without dividing an individual apple.

20. A hare is 50 leaps before a greyhound, and takes 4 leaps to the greyhound 3, but two leaps of the hound are equal to 3 of the hare's. How many leaps must the greyhound take before he catches the hare?

SOLUTION.

Since 2 leaps of the greyhound equals 3 leaps of the hare, it follows that 6 leaps of the greyhound equals 9 leaps of the hare.

But while the greyhound takes 6 leaps, the hare takes 8 leaps; therefore, while the hare takes 8 leaps, the greyhound gains on her 1 leap. Hence, to gain 50 leaps she must take $50\times8=400$ leaps; but while the hare takes 400 leaps, the greyhound would take 300 leaps, since the number of leaps taken by them were as 4 to 3.

21. From a cask of wine a tenth part is drawn out, and then the cask is filled with water; after which, a tenth part of the mixture is drawn out; again, the cask is filled, and again a tenth part of the mixture is drawn out. Now suppose the fluids mix uniformly at each time the cask is replenished, what fractional part of pure wine will remain after the process of drawing out, and replenishing has been repeated ten times?

SOLUTION.

Since $\frac{1}{10}$ of the wine is drawn out at the first drawing, there must remain $\frac{9}{10}$. And after the cask is filled with water, $\frac{1}{10}$ of the whole being drawn out, there will remain $\frac{9}{10}$ of *the* mixture; but $\frac{9}{10}$ of the mixture, we have already seen, is pure wine; therefore after the second drawing there will remain $\frac{9}{10}$ of $\frac{9}{10}$ of pure wine, or $\frac{9^2}{10^2}$. By similar reasoning, we see that after the third drawing there will remain $\frac{9}{10}$ of $\frac{9}{10}$ of $\frac{9}{10}$ of pure wine, or $\frac{9^3}{10^3}$.

From this, we see that the part of pure wine remaining is expressed by the ratio $\frac{9}{10}$, raised to a power whose exponent is the number of times the cask has been drawn from. Hence, in the present question, the fractional part of pure wine is $\frac{9^{10}}{10^{10}}=\frac{3486784401}{10000000000}=0.3486784401$, which is nearly 35 per cent.

22. Suppose, from an acorn, there shoots up a single stalk at the end of the year; that at the end of each year thereafter, this stalk puts forth as many new branches as it is years old: also suppose all the branches to follow the same law, that is, to produce as many new branches as they are years old. How many branches will this oak tree consist of at the end of 20 years?

SOLUTION.

From the conditions of the question, we know that at the end of the *first* year there will be simply one stalk or branch, which we will denote by 1_0; at the end of the *second* year, this branch will become one year old, and produce a new branch, so that we shall have 1_1+1_0; at the end of the *third* year the branches 1_1+1_0 will become 1_2+1_1, the first of which being 2 years of age, will produce two new branches, the other will produce one new one; we shall therefore have $1_2+1_1+3_0$. Proceeding in this way, we obtain the following results:

End of	1st	year	we have	1_0	$=$	1
"	2	"	"	1_1+1_0	$=$	2
"	3	"	"	$1_2+1_1+3_0$	$=$	5
"	4	"	"	$1_3+1_2+3_1+8_0$	$=$	13
"	5	"	"	$1_4+1_3+3_2+8_1+21_0$	$=$	34
"	6	"	"	$1_5+1_4+3_3+8_2+21_1+55_0$	$=$	89

In this scheme, the small figures at the bottom of the larger ones, denote the age in years, of the branches to which they are attached. Thus, at the end of the fifth year, there will be one branch 4 years old, one branch 3 years old, three branches 2 years old, eight branches 1 year old, and twenty-one new branches of no age.

The law of the above series is obvious. It is such, *that twice any term, increased by the sum of all the preceding terms, gives the next succeeding term.*

These terms may be found most easily by continual addition, as given on the following page, where each succeeding term is found by adding the two preceding ones.

	0	new branches 1st year,
total branches 1st year,	1	
	1	new branches 2d year,
total branches 2d year,	2	
	3	new branches 3d year,
total branches 3d year,	5	
	8	new branches 4th year,
total branches 4th year,	13	
	21	new branches 5th year,
total branches 5th year,	34	
	55	new branches 6th year,
total branches 6th year,	89	
	144	new branches 7th year,
total branches 7th year,	233	
	377	new branches 8th year,
total branches 8th year,	610	
	987	new branches 9th year,
total branches 9th year,	1597	
	2584	new branches 10th year,
total branches 10th year,	4181	
	6765	new branches 11th year,
total branches 11th year,	10946	
	17711	new branches 12th year,
total branches 12th year,	28657	
	46368	new branches 13th year,
total branches 13th year,	75025	
	121393	new branches 14th year,
total branches 14th year,	196418	
	317811	new branches 15th year,
total branches 15th year,	514229	
	832040	new branches 16th year,
total branches 16th year,	1346269	
	2178309	new branches 17th year,
total branches 17th year,	3524578	
	5702887	new branches 18th year,
total branches 18th year,	9227465	
	14930352	new branches 19th year,
total branches 19th year,	24157817	
	39088169	new branches 20th year,
total branches 20th year.	63245986	

CHAPTER XVI.

MISCELLANEOUS QUESTIONS.

93. What are the prime factors of 2006?

2. What are the prime factors of 3742?

3. What is the greatest common measure of 720, 360, and 180?

4. What is the greatest common measure of 420, 147, and 210?

5. What is the least common multiple of 4, 16, 24, and 40?

6. What is the least common multiple of 8, 36, and 100?

7. What are all the divisors of 376?

8. What are all the divisors of 23456?

9. What is the sum of the divisors of 7866?

10. What is the sum of the divisors of 1000?

11. Reduce $\frac{378}{456}$ to its lowest terms.

12. Reduce $\frac{123456}{234568}$ to its lowest terms.

13. Reduce the improper fraction $\frac{1236}{63}$ to a mixed fraction.

14. Reduce $\frac{4567}{756}$ to a mixed fraction.

15. Reduce $67\frac{5}{11}$ to an improper fraction.

16. Reduce $37\frac{7}{3}$ to an improper fraction.

17. What is the product of $\frac{365}{21}$ into $\frac{35}{17}$?

18. What is the product of $\frac{29}{36}$ into $\frac{109}{58}$?

19. Reduce the compound fraction $\frac{1}{2}$ of $\frac{3}{8}$ of $\frac{4}{7}$ of $\frac{14}{25}$ to a simple fraction.

20. Reduce $\frac{3}{11}$, $\frac{4}{33}$, and $\frac{6}{7}$ to fractions having a common denominator.

21. What is the sum of $\frac{4}{15}$, $\frac{6}{7}$, and $\frac{3}{10}$?

22. What is the quotient of $\frac{37}{240}$ divided by $\frac{3}{8}$?

23. Reduce the complex fraction $\dfrac{4\frac{1}{7}}{8}$ to its simplest form.

24. What is the value of $\frac{3}{17}$ of a mile?

25. Can the vulgar fraction $\frac{3}{11}$ be accurately expressed in decimals?

26. How many places of decimals will be required to express $\frac{13}{32}$?

27. Find the compound repetend equivalent to $\frac{1}{28}$.

28. Find the perfect repetend arising from $\frac{1}{47}$.

29. Convert 0.3756 into a vulgar fraction.

30. Convert $\frac{371}{629}$ into a continued fraction.

31. Find some of the approximative values of the continued fraction

$$\cfrac{1}{3+\cfrac{1}{3+\cfrac{1}{3+\cfrac{1}{3+\&c.}}}}$$

32. What must be the length of a thread, which will wind spirally about a cylinder of 4 feet in circumference, and 60 feet in length, the distance between each turn of the thread, being 1 foot?

33. Required to divide the number 90 into four parts; such that, if the first be increased by 5, the second decreased by 4 the third multiplied by 3, and the fourth divided by 2, the results, in each case, shall be the same.

34. If A can perform a piece of work in 10 days, B in 12 days, C in 16 days, then how many days will be required for all together to perform the work?

35. A shepherd, in the time of war, fell in with a party of soldiers who plundered him of half his flock, and half a sheep over; afterwards a second party met him, who took half of what he had left, and half a sheep over; and soon after this, a third party met him, and used him in the same manner; and then he had only five sheep left. It is required to find what number of sheep he had at first.

36. Four persons, A, B, C, and D, spent 20 shillings in company; when A proposed to pay $\frac{1}{3}$, B $\frac{1}{4}$, C $\frac{1}{5}$, and D $\frac{1}{6}$ part; but when the money came to be collected, they found it was not sufficient to answer the intended purpose. The question then is, to find how much each person must contribute to make up the whole reckoning, supposing their several shares to be to each other in the proportion above specified.

37. If $\frac{3}{7}$ of a pound of cinnamon is worth 18 cents, what will 7 $\frac{1}{8}$ pounds cost?

38. If a family of 8 persons spend $480 in 32 months, how much would 16 persons spend in 8 months?

39. Two persons depart from the same place at the same time; the one travels 30, the other 35 miles a day. How far distant are they after 7 days, if they travel both in the same direction; and how far if they travel in contrary directions?

40. A stationer sold quills at 10*s*. 6*d*. a thousand, by which he cleared $\frac{1}{3}$ of his money; but growing scarce, he raised them to 12*s*. a thousand. What did he clear per cent. by the latter price?

41. How much can a person give for an annuity of $400, which has to run 12 years, if the interest be reckoned at 3 per cent.? Ans. $3981.602.

42. Eight oxen have, in seven weeks, eaten all the grass which grew on 400 square rods of land, in such a manner that they not only ate all the grass which at first was there, but also that which grew during the time they were grazing. In like manner, have 9 oxen, in eight weeks, eaten up all the grass upon 500 square rods of land. How many oxen can in this way, graze, for 12 weeks, upon 600 square rods of land? Ans. 8.

43. A and B possess, together, only $\frac{2}{3}$ of the property of C; B and C have, together, 6 times as much as A; were B \$680 richer than he actually is, then he would have as much as A and C together. How much has each?

Ans. A has \$200, B \$360, and C \$840.

44. How often can the 26 letters of the alphabet be transposed? Ans. 403291461126605635584000000 times.

NOTE.—All the inhabitants of the globe, taken together, could not, in a thousand millions of years, write out all these transpositions of the 26 letters, even supposing that each wrote 40 pages daily, each of which pages should contain 40 different transpositions of the letters.

45. A debt, due at this present time, amounting to \$1200, is to be discharged in seven yearly and equal payments. What is the amount of one of these payments, if the interest be calculated at 4 per cent.? Ans. \$199.931.

46. A usurer lent a person \$600, and drew up for the amount a bond of \$800, payable in 3 years, bearing no interest. What did he take per cent., if compound interest be taken into consideration? Ans. 10.06424 per cent.

47. A person being asked about his salary, answered, "At present I have \$550; but when I first entered into office I had no more than \$100; but, on account of my industry, I received, each year, an addition of \$30 to my income. How long was he engaged? Ans. 16 years.

48. A debtor being unable to pay his debt, amounting to \$12950, at once, agrees with his creditors to discharge it by monthly instalments, viz: \$600 the first month, and each succeeding month \$50 more than the preceding one. In how many months will he have discharged his whole debt? And how much does he pay the last month?

Ans. In 14 months, and \$1250.

49. A person dying leaves half of his property to his wife, one-sixth to each of two daughters, one-twelfth to a servant, and the remaining \$600 to the poor. What was the amount of his property? Ans. \$7200.

50. We know, from natural philosophy, that any body, which falls *in vacuo*, passes, in the first second, through a space of $16\frac{1}{12}$ feet; and in each succeeding second, $32\frac{1}{6}$ feet more than in the one immediately preceding. Now, if a body has been falling 20 seconds, how many feet will it have fallen the last second? And how many in the whole time?

Ans. $627\frac{1}{4}$ feet, and $6433\frac{1}{3}$ feet.

51. An estate of \$7500 is to be divided between a widow, two sons, and three daughters, so that each son shall receive twice as much as each daughter, and the widow herself \$500 more than all the children. What was her share? And what the share of each child?

Ans. Widow's share \$4000.
Each son's \$1000.
Each daughter's \$500.

52. Three soldiers, in a battle, make \$96 booty, which they wish to share equally. In order to do this, A, who made the most, gives B and C as much as they already had; in the same manner, B then divided with A and C, and after this, C with A and B. If, then, by these means, the intended equal division is effected, how much booty did each soldier make?

Ans. A \$52, B \$28, and C \$16.

53. A purse of $2850 is to be divided among three persons, A, B, and C. A's share is to be to B's as 6 to 11, and C is to have $300 more than A and B together. What is each one's share?

Ans. A's $450, B's $825, C's $1575.

54. A father leaves a number of children, and a certain sum, which they are to divide amongst them as follows: The first is to receive $100, and then the 10th part of the remainder; after this, the second has $200, and the 10th part of the residue; again, the third receives $300, and the 10th part of the remainder; and so on, each succeeding child is to receive $100 more than the one immediately preceding, and then the 10th part of that which still remains. At last it is found that all the children have received the same. What was the fortune left? And how many children were there?

Ans. { The fortune left was $8100, and the number of children 9.

55. Two carpenters, 24 journeymen, and 8 apprentices, received at the end of a certain time $144. The carpenters received $1 per day, each journeyman half a dollar, and each apprentice 25 cents. How many days were they employed?

Ans. 9 days.

56. A man, to please his children, brings home a number of apples, and divides them as follows: To the first and eldest of his children he gives the half of the whole number, less 8; to the second, the half of the remainder, again diminished by 8; and he does the same with the third and fourth. After this he gives the 20 remaining apples to the fifth. How many apples did he bring home?

Ans. 80.

57. A farmer being asked how many sheep he had, answered that he had them in five fields; in the first he had $\frac{1}{4}$, in the 2d $\frac{1}{6}$, in the 3d $\frac{1}{8}$, and in the 4th $\frac{1}{12}$, and in the 5th 450. How many had he?

Ans. 1200.

58. I once had an untold sum of money lying before me. From this I first took away the 3d part, and put in its stead \$50. A short time after I took from the sum thus augmented, the 4th part, and put again in its stead \$70. I then counted my money and found \$120. What was the original sum?

Ans. \$25.

59. After paying $\frac{1}{4}$ and $\frac{1}{5}$ of my money, I had 66 guineas left in my purse. How many guineas were in it at first?

Ans. 120.

60. A countryman brings his eggs to market, and first sells 4 more than the half of them, then he goes further, and sells half of the remainder, and 2 over. Now, 6 eggs more than half of the remainder are stolen from him, and dissatisfied about this loss, he returns to his village with the two eggs which remained in his basket. How many eggs did he take to town?

Ans. 80.

61. A person goes to a tavern with a certain sum of money in his pocket, where he spends 2 shillings; he then borrows as much money as he had left, and going to another tavern, he there spends 2 shillings also; then borrowing again as much money as was left, he went to a third tavern, where likewise he spent 2 shillings and borrowed as much as he had left; and again spending 2 shillings at a fourth tavern, he then had nothing remaining. What had he at first?

Ans. 3*s.* 9*d.*

62. A cistern can be filled by three pipes; by the first in $1\frac{1}{3}$ hours, by the second in $3\frac{1}{3}$ hours, and by the third in 5 hours. In what time will this cistern be filled when all three pipes are open at once?

Ans. In 48 minutes.

63. To divide the number 36 into 3 such parts that $\frac{1}{2}$ of the first, $\frac{1}{3}$ of the second, and $\frac{1}{4}$ of the third, may be all equal to each other.

Ans. 8, 12, and 16.

64. A person possesses a wagon with a mechanical contrivance, by which the difference of the number of revolutions of the wheels on a journey may be determined. It is known that each of the fore-wheels is $5\frac{1}{4}$, and that each of the hind wheels is $7\frac{1}{8}$ feet in circumference. Now, when in a journey the fore-wheel has made 2000 revolutions more than the hind-wheel, how great was the distance traveled?

Ans. 39900 feet.

65. A dog pursues a hare. Before the dog started, the hare had made 50 paces, and this is the distance between them at first. The hare takes 6 paces to the dog's 5; and 9 of the hare's paces are equal to 7 of the dog's. How many paces can the hare take before the dog overtakes her? Ans. 700.

66. A person wishes to dispose of his horse by lottery. If he sells the tickets at \$2 each, he will lose \$30 on his horse; but if he sells them at \$3 each, he will receive \$30 more than his horse cost him. What is the value of the horse, and the number of tickets? Horse worth \$150, No. tickets, 60.

67. The four following numbers,

20809130829564551426 36, 49378013475106807329 48,

72628104764100161630 52, 21497210869324158934 0948,

when added together, by taking two at a time, produce six distinct sums; each of which is a *perfect cube*. What are the six roots of these cube numbers?

Ans. $\begin{cases} 19146344,\ 21062342,\ 60097344, \\ 23021160,\ 60359866,\ 60571840. \end{cases}$

68. What is the square of 12890625?

Ans. 166168212890625.

NOTE.—In this question it will be observed that the square of the above number ends with the same set of figures as the number itself; and this must hold good for any power of the above number.

69. Suppose \$1 had been put out, at compound interest, at 7 per cent., the 5th day of October, 1585. How much would it have amounted to on the 1st of December, 1841?

Ans. \$3364085.041.

70. A gentleman hires a servant, and promises him, for the first year, only \$60 in wages, but for each following year \$4 more than for the preceding. How much will the servant receive for the 17th year of his engagement? and how much for all 17 years together? Ans. \$124, and \$1564.

71. A person had two barrels, and a certain quantity of wine in each. In order to have an equal quantity in each, he poured out as much of the first cask into the second, as it already contained; then, again, he poured out as much of the second into the first as it then contained, and lastly he poured out again, as much from the first into the second, as there was still remaining in it. At last he had 16 gallons of wine in each cask. How many gallons did they contain originally? Ans. The first 22, the second 10 gallons.

72. What is the sum of the cubes of the seven following fractions: $\frac{429}{525}$, $\frac{497}{525}$, $\frac{504}{525}$, $\frac{510}{525}$, $\frac{511}{525}$, $\frac{513}{525}$, and $\frac{516}{525}$?

Ans. 6.

73. A general, wishing to draw up his regiment into a square, tried it in two ways. The first time he had 39 men over; the second, having extended the side of the square by one man, he wanted 50 men to complete the square. What was the number of soldiers in the regiment?

Ans. 1975 men.

74. From a sum of money, \$50 more than the half of it is first taken away; from the remainder \$30 more than its fifth part; and again, from the 2d remainder \$20 more than its fourth part. At last there remained only \$10. What was the original sum? Ans, \$275.

75. What is the sum of the cubes of the five following fractions: $\frac{349}{345}$, $\frac{347}{345}$, $\frac{369}{345}$, $\frac{379}{345}$, and $\frac{393}{345}$? Ans. 6.

76. The siderial revolution of Venus is 224.700817 days. How can this period be represented by smaller numbers?

Ans. By $\frac{224}{1}$, $\frac{225}{1}$, $\frac{674}{3}$, $\frac{1573}{7}$, $\frac{2247}{10}$, $\frac{26290}{117}$, &c.

77. Find ten numbers, such that the first with $\frac{1}{5}$ of all the rest shall make 845693; the second with $\frac{1}{6}$ of all the rest shall make 845693; the third with $\frac{1}{7}$ of all the rest shall make 845693; the fourth with $\frac{1}{8}$, the fifth with $\frac{1}{9}$, the sixth with $\frac{1}{10}$, the seventh with $\frac{1}{11}$, the eighth with $\frac{1}{12}$, the ninth with $\frac{1}{13}$, and the tenth with $\frac{1}{14}$ of all the rest, shall make *respectively* the same number, 845693.

Ans. { 34883, 197045, 305153, 382373, 440288, 485333, 521369, 550853, 575423, 596213.

ERRATA.

Question 3d, on page 226, ought to read as follows: "In how many ways may the first 25 letters of the alphabet be read in this way?"

Question 9th, on page 230, ought to read as follows: "A and B can, together, do a piece of work in 8 days; A and C can, together, do it in 9 days; and B and C can, together, do it in 10 days. How many days would it require for each to perform the work alone?"

There are several other slight errors, which are considered too trifling to need correction in this place.